1. Some facts about Chambers, Weyl group etc

Let R be a reduced root system in V with Weyl group W. Let $(\cdot \mid \cdot)$ be a W-invariant inner product on V. Given $\alpha \in R$, let $L_\alpha := \{\alpha\}^\perp$ (with respect to the given inner product); this is the hyperplane of V fixed by the reflection s_α. It is easily seen that $L_{w\alpha} = w(L_\alpha)$ for all $w \in W, \alpha \in R$. Thus W permutes the set $\{L_\alpha : \alpha \in R\}$. Define $V^{reg} := V \setminus \cup_{\alpha \in R} L_\alpha$. The connected components of V^{reg} are called *chambers* of R.

Each $x \in V^{reg}$ determines a partition of R into *positive* and *negative* roots as follows:

$$R^+(x) := \{\alpha \in R : (\alpha \mid x) > 0\} \text{ and } R^-(x) := \{\alpha \in R : (\alpha \mid x) < 0\}$$

Exercise 1: Prove that for $x, y \in V^{reg}$, $R^+(x) = R^+(y) \iff x, y$ belong to the same chamber. Thus if C is a chamber, we can define $R^\pm(C)$ to be $R^\pm(x)$ where x is any element of C. Clearly, $R^-(C) = -R^+(C)$.

Now, fix a chamber C, and consider the set $R^+(C)$. An element $\alpha \in R^+(C)$ is said to be *decomposable* if $\exists \beta_1, \beta_2 \in R^+(C)$ such that $\alpha = \beta_1 + \beta_2$; it is *indecomposable* if it is not decomposable. Let $B(C)$ denote the set of indecomposable roots in $R^+(C)$; the elements of $B(C)$ are called *simple roots.*

Theorem 1. (1) *$B(C)$ is a basis of V.*
(2) $R^+(C) \subset \mathbb{Z}_{\geq 0}(B(C))$

Proof: Fix $x \in C$, and define a function $h : R^+(C) \to \mathbb{R}_{>0}$ by $h(\alpha) := (\alpha \mid x)$. Let $m > 0$ be the minimum value of h; thus $h(\alpha) \geq m$ for all $\alpha \in R^+(C)$. We first prove (2); given $\beta \in R^+(C)$, if it is indecomposable, we are done, else write $\beta = \gamma_1 + \gamma_2$ where $\gamma_i \in R^+(C)$. Observe $h(\gamma_i) \leq h(\beta) - m$ for $i = 1, 2$. If both γ_i are indecomposable, we are done, else continue this process. This process must be finite, since the value of h decreases at least by $m > 0$ at each step. This completes the proof of (2), and also shows that $B(C)$ spans V.

Next observe, $\alpha, \beta \in B(C)$ implies that $(\alpha \mid \beta) \leq 0$. If not, then $n_{\alpha,\beta} > 0$ and by one of our earlier lemmas, $\gamma := \alpha - \beta$ would be a root. If $\gamma \in R^+(C)$, then $\alpha = \beta + \gamma$, which contradicts the indecomposability of α; if on the other hand, $-\gamma \in R^+(C)$, one similarly has $\beta = \alpha + (-\gamma)$. The linear independence of $B(C)$ now follows from the following Lemma.

Lemma 1. *Let V be a finite dimensional vector space with inner prduct $(|)$. Suppose $S \subset V$ satisfies (i) all elements of S lie on the same side of some hyperplane of V, and (ii) $(v \mid w) \leq 0$ for all $v, w \in S$. Then S is linearly independent.*

This is just Lemma 3 on page 82 of Bourbaki (*Lie Groups and Lie Algebras, Chaps IV-VI*). With notation as above, we have:

Proposition 1. (1) $C = \{\gamma \in V : (\gamma \mid \alpha) > 0 \, \forall \alpha \in B(C)\}$.
(2) *If $\alpha \in B(C)$, $s_\alpha(R^+(C) \setminus \{\alpha\}) = R^+(C) \setminus \{\alpha\}$.*
(3) *W is generated by $\{s_\alpha : \alpha \in B(C)\}$.*
(4) *W acts simply transitively on the set of chambers.*

Proof: (1): follows from Exercise 1 above.

(2): Let $\beta \in R^+(C)$; write $\beta = \sum_{\gamma \in B(C)} c_\gamma \gamma$ with $c_\gamma \in \mathbb{Z}_{\geq 0} \forall \gamma$. Since R is reduced, if $\beta \neq \alpha$, then $\exists \gamma_0 \in B(C) \setminus \{\alpha\}$ such that $c_{\gamma_0} > 0$. Now $s_\alpha(\beta) = \beta - n_{\beta\alpha}\alpha = \sum_{\gamma \neq \alpha} c_\gamma \gamma + (c_\alpha - n_{\beta\alpha})\alpha$; thus when expressed as a linear combination of elements of $B(C)$, $s_\alpha(\beta)$ has at least one positive coeffient, namely that of γ_0. Part (2) of theorem 1 implies that $s_\alpha(\beta) \in R^+(C)$. This completes the proof.

(3): Let $W' := \langle s_\alpha : \alpha \in B(C) \}$. *Claim:* Given $\beta \in R^+(C)$, there is a $w \in W'$ such that $w\beta = \alpha \in B(C)$. Assuming the claim, we would have $s_\beta = w^{-1} s_\alpha w$ which clearly belongs to W', thereby proving (3). Suppose the claim were false, it follows from (2) that $W'\beta \subset$

$R^+(C) \backslash B(C)$. For a root $\epsilon = \sum_{\gamma \in B(C)} c_\gamma \gamma$, define $\mathrm{ht}(\epsilon) := \sum c_\gamma$. Let $\beta' = \sum_{\gamma \in B(C)} d_\gamma \gamma$ be an element of minimal height in $W'\beta$. The height minimality condition in particular implies that $\mathrm{ht}(s_\gamma \beta) \geq \mathrm{ht}(\beta) \, \forall \gamma \in B(C)$; this in turn implies that $(\beta \mid \gamma) \leq 0$ for all simple roots γ. But then, $(\beta' \mid \beta') = \sum_\gamma d_\gamma (\beta' \mid \gamma) \leq 0$, a contradiction.

(4) Given $x \in V^{reg}$, let λ be an element of maximal height in the W-orbit of x. By arguments similar to (3), it is clear that $(\lambda \mid \alpha) \geq 0$ for all $\alpha \in B(C)$. Equality cannot occur for any α, since that would mean λ lies on the hyperplane L_α, and therefore that x is not in V^{reg}. Thus $\lambda \in C$. This proves that W acts transitively on the set of chambers. To prove that it acts *simply* one requires a few more notions. Each $w \in W$ can be written as $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ where each $\alpha_i \in B(C)$. The length k of the shortest such word is defined to be the length of w. It is a fact that *the length of an element $w \in W$ is the number of positive roots α for which $w\alpha$ is a negative root.* Now observe that if $w \in W$ is such that $wC = C$, we also have $w(R^+(C)) = R^+(C)$. The preceding sentence then implies that the length of w must be zero, i.e, $w = 1$. $\square$