We assume all Lie algebras and vector spaces are finite dimensional over k unless otherwise mentioned.

1. Cartan's criterion, Killing form, semisimplicity

If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V, then $\mathfrak{g}$ is said to be a **linear Lie algebra**.

Theorem 1. *(Lie's theorem) Let k be an algebraically closed field of characteristic zero, and let V be a nonzero finite dimensional vector space over k. Suppose $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a solvable Lie algebra. Then there is a nonzero vector $v \in V$ such that $Xv = \lambda(X)\, v$ for all $X \in \mathfrak{g}$ (i.e a common eigenvector).*

Note that λ in the statement above must be a linear functional on $\mathfrak{g}$. Recall that similar results have been encountered before: (a) If $\mathfrak{g}$ is an abelian Lie subalgebra of $\mathfrak{gl}(V)$ such that every $X \in \mathfrak{g}$ is diagonalizable, then a simultaneous eigenvector exists (linear algebra), (b) If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ such that every $X \in \mathfrak{g}$ is a nilpotent operator on V, then $\exists$ a common eigenvector for all $X \in \mathfrak{g}$ (with eigenvalue necessarily zero). This was encountered in the context of Engel's theorem.

A slightly more general version of Lie's theorem is the following: With the same hypothesis on k, if (ϕ, V) is a representation of a solvable Lie algebra $\mathfrak{g}$, then $\exists v \neq 0$ in V such that $X \cdot v = \lambda(X) v \, \forall X \in \mathfrak{g}$, for some $\lambda \in \mathfrak{g}^*$. This can be obtained by applying Lie's theorem to the solvable Lie algebra $\phi(\mathfrak{g})$.

Remarks: Assume the hypotheses of Theorem 1. Then we have :

1. The operators in $\mathfrak{g}$ can in fact be simultaneously upper triangularized. In other words, $\exists$ a basis $\mathcal{B}$ of V such that $[X]_{\mathcal{B}}$ is upper triangular for all $X \in \mathfrak{g}$. This can be proved by repeated application of the general version of Lie's theorem to suitable quotients of V.
2. $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra. This follows from the simultaneous upper triangularization above; the commutator subalgebra would become a subalgebra of the Lie algebra of strictly upper triangular matrices, which is nilpotent.
3. This also gives us that if $X \in \mathfrak{g}$ and $Y \in [\mathfrak{g}, \mathfrak{g}]$, then $\operatorname{tr}(XY) = 0$.

It turns out that the converse of the last statement above is true, and gives a powerful criterion for solvability of a linear Lie algebra.

Theorem 2. *(Cartan's criterion) Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra. If $\operatorname{tr}(XY) = 0 \, \forall X \in \mathfrak{g}, Y \in [\mathfrak{g}, \mathfrak{g}]$, then g is solvable.*

We also have the following easy extension of this result to not necessarily linear Lie algebras:

Proposition 1. *If (ϕ, V) is a representation of $\mathfrak{g}$ such that $\operatorname{tr}(\phi(X)\phi(Y)) = 0$ for all $X \in \mathfrak{g}, Y \in [\mathfrak{g}, \mathfrak{g}]$, and if $\ker \phi$ is solvable, then $\mathfrak{g}$ is solvable.*

Note that $\phi = \operatorname{ad}$ gives a ready example of a representation whose kernel ($= Z(\mathfrak{g})$) is solvable. This motivates the following definition:

Definition 1. The Killing form of the Lie algebra $\mathfrak{g}$ is defined to be the symmetric bilinear form $B(X, Y) := \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y : \mathfrak{g} \to \mathfrak{g})$.

The above discussion shows that if $B(X, Y) = 0 \, \forall X \in \mathfrak{g}, Y \in [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}$ is solvable. The Killing form has the following important properties:

1. ($\mathfrak{g}$-invariance) $B([X, Y], Z) + B(Y, [X, Z]) = 0$, $\forall X, Y, Z \in \mathfrak{g}$.
2. If I is an ideal of $\mathfrak{g}$, let B_I denote the Killing form of I viewed as a Lie algebra in its own right. Then $B_I(X, Y) = B(X, Y)$ for all $X, Y \in I$. In other words, $B_I = B|_{I \times I}$.

(3) If I is an ideal of $\mathfrak{g}$, then so is $I^\perp := \{X \in \mathfrak{g} : B(X,Y) = 0\, \forall Y \in I\}$.

Define the ideal $\ker B := \mathfrak{g}^\perp$. Recall that one would call the symmetric bilinear form B *nondegenerate* if $\ker B = 0$. The following theorem gives a very useful criterion for semisimplicity of $\mathfrak{g}$.

Theorem 3. *A Lie algebra $\mathfrak{g}$ is semisimple $\iff$ its Killing form B is nondegenerate.*

Proof: Letting $\mathfrak{k} := \ker B$, we have $B(X,Y) = 0$ for all $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$, in particular for all $Y \in [\mathfrak{k}, \mathfrak{k}]$. Since the Killing form of $\mathfrak{k}$ is the restriction of B to $\mathfrak{k} \times \mathfrak{k}$, Cartan's criterion shows that $\mathfrak{k}$ is solvable. For the converse, recall that if $\mathfrak{g}$ has a nonzero solvable ideal R, then it also has a nonzero abelian ideal I (take I to be the last nonzero term in the derived series of R). It is easy to see that $\operatorname{ad} X \operatorname{ad} Y \operatorname{ad} X = 0$ for all $X \in I$ and $Y \in \mathfrak{g}$. Thus $\operatorname{ad} X \operatorname{ad} Y$ is a nilpotent operator, and must have trace 0. Thus $B(X,Y) = 0$ for all $X \in I, Y \in \mathfrak{g}$, contradicting the nondegeneracy of B. □

Corollary 1. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ (or any algebraically closed field of characteristic 0) and let I be an ideal of $\mathfrak{g}$. Then*

$$\mathfrak{g} = I \oplus I^\perp$$

Proof: From general facts about nondegenerate bilinear forms, it follows that $\dim \mathfrak{g} = \dim I + \dim I^\perp$. It remains only to show that $\mathfrak{k} := I \cap I^\perp$ is zero. But clearly $B(X,Y) = 0\, \forall X, Y \in \mathfrak{k}$. Hence Cartan's criterion shows $\mathfrak{k}$ is a solvable ideal of $\mathfrak{g}$, and hence 0. □

Corollary 2. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\exists$* simple *ideals $\mathfrak{g}_i$ $(i = 1, \cdots, r)$ such that*

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$$

Further, the $\mathfrak{g}_i$ are unique.

The proof is a straightforward induction argument that uses corollary 1.

2. Abstract Jordan decomposition

Let V be a finite dimensional vector space over the algebraically closed field k. The following is the *abstract Jordan decomposition* theorem for linear operators on V.

Theorem 4. *Let T be a linear operator on V. There exist unique linear operators T_s and T_n such that (a) $T = T_s + T_n$, (b) T_s is diagonalizable and T_n is nilpotent and (c) $[T_s, T_n] = 0$.*

In fact, it is further true that T_s and T_n are polynomials in T. In terms of matrices, once the (usual) Jordan matrix form of T is found, T_s is just the matrix of diagonal entries of this Jordan form, and $T_n = T - T_s$ is the matrix consisting of ones and zeros on the first superdiagonal, and zeros elsewhere.

Next, suppose that $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a linear Lie algebra. If $X \in \mathfrak{g}$, it is not necessarily true that X_s and X_n must also lie in $\mathfrak{g}$ (for instance, take $\mathfrak{g}$ to be the one dimensional Lie subalgebra spanned by a non-diagonalizable, non-nilpotent operator). It is thus remarkable that the following theorem is true.

Theorem 5. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semisimple Lie algebra. Then $X \in \mathfrak{g} \Rightarrow X_s, X_n \in \mathfrak{g}$.*

Thus elements of a semisimple linear Lie algebra admit an abstract Jordan decomposition. This can be extended to all semisimple Lie algebras via the map $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$; since $\mathfrak{g}$ is semisimple, $\ker \operatorname{ad} = 0$. Thus $\tilde{\mathfrak{g}} \cong \mathfrak{g}$ is a semisimple linear Lie algebra. Applying the theorem above to $\tilde{\mathfrak{g}}$, we deduce: Given $X \in \mathfrak{g}$, there exist unique elements $X_s, X_n \in \mathfrak{g}$ such that (a) $X = X_s + X_n$, (b) $\operatorname{ad} X_s$ is diagonalizable and $\operatorname{ad} X_n$ is nilpotent and (c) $[X_s, X_n] = 0$. In fact,

$\operatorname{ad} X_s$ and $\operatorname{ad} X_n$ are just the diagonalizable and nilpotent parts of $\operatorname{ad} X$. The decomposition $X = X_s + X_n$ is called the abstract Jordan decomposition of X in $\mathfrak{g}$. When $\mathfrak{g}$ is a semisimple *linear* Lie algebra, the two notions of abstract Jordan decomposition defined above can be easily checked to coincide.

The usefulness of this notion arises from the following important theorem.

Theorem 6. *(Preservation of Jordan decomposition) Let $\mathfrak{g}$ be a semisimple Lie algebra and let (ϕ, V) be a representation of $\mathfrak{g}$. Then*

$$\phi(X_s) = \phi(X)_s \text{ and } \phi(X_n) = \phi(X)_n$$

In other words, X_s and X_n intrinsically keep track of the diagonalizable and nilpotent parts of the action of X in *every representation* of $\mathfrak{g}$.

Definition 2. Let $\mathfrak{g}$ be a semisimple Lie algebra. An element $X \in \mathfrak{g}$ is called *semisimple* (resp. *nilpotent*) if $X = X_s$ (resp. $X = X_n$).

We observe that $\mathfrak{g}$ must contain at least one nonzero semisimple element. Otherwise all $X \in \mathfrak{g}$ would be ad-nilpotent; Engel's theorem would then imply that $\mathfrak{g}$ is nilpotent, hence solvable, contradicting its semisimplicity. Semisimple elements act as diagonalizable operators in all representations of $\mathfrak{g}$ and nilpotent elements act as nilpotent operators in all representations of $\mathfrak{g}$.

Examples: Let $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$. Recall the standard basis elements $H := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $X := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $Y := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. The element H is semisimple while X and Y are nilpotent. Similarly for $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, the diagonal trace 0 matrices are all semisimple, while the matrix units E_{ij} for $i \neq j$ (with a 1 in the $(i,j)^{th}$ entry and zeros elsewhere) are all nilpotent.

Definition 3. Let g be a semisimple Lie algebra. A Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is called a *toral subalgebra* if all its elements are semisimple (i.e $X = X_s$ for all $X \in \mathfrak{t}$).

Lemma 1. *If $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{g}$, then $\mathfrak{t}$ is abelian.*

Proof: Let $X \in \mathfrak{t}$; then $\operatorname{ad} X$ is diagonalizable, and leaves $\mathfrak{t}$ invariant. Thus $\operatorname{ad} X|_{\mathfrak{t}}$ is also diagonalizable. We thus only need to show that all eigenvalues of $\operatorname{ad} X|_{\mathfrak{t}}$ are zero. If not, let $\lambda \neq 0$ be an eigenvalue, with eigenvector $Y \in \mathfrak{t}$. Thus $\operatorname{ad} X(Y) = \lambda Y$. This implies that X and Y are linearly independent; we also have $\operatorname{ad} Y(X) = -\lambda Y$. Thus $\operatorname{ad} Y$ leaves the span of X, Y invariant, and must therefore act diagonalizably on this 2 dimensional subspace. With respect to the basis X, Y, the matrix of $\operatorname{ad} Y$ is $\left[\begin{smallmatrix} 0 & 0 \\ -\lambda & 0 \end{smallmatrix}\right]$, which is clearly not diagonalizable. □

We now let $\mathfrak{h}$ denote a maximal toral subalgebra of $\mathfrak{g}$. Since nonzero semisimple elements exist, there are toral subalgebras of dim 1; $\mathfrak{h}$ is thus nonzero. Observe that the maximality of $\mathfrak{h}$ implies that if $X \in \mathfrak{g}$ is semisimple and commutes with all elements of $\mathfrak{h}$, then $X \in \mathfrak{h}$. The following theorem asserts that the same is true for *all elements* of $\mathfrak{g}$ (not just for the semisimple ones).

Theorem 7. *Let $\mathfrak{h}$ be a maximal toral subalgebra of the semisimple Lie algebra $\mathfrak{g}$. Then $\mathrm{C}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ where the centralizer $\mathrm{C}_{\mathfrak{g}}(\mathfrak{h}) := \{X \in \mathfrak{g} : [X, H] = 0 \, \forall H \in \mathfrak{h}\}$.*