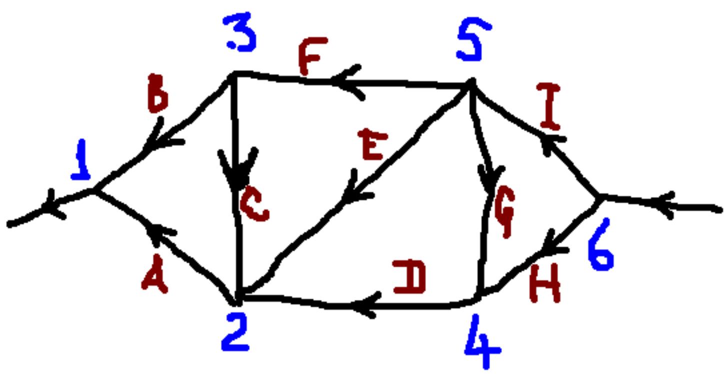
AN APPLICATION OF LINEAR ALGEBRA TO NETWORKS

K. N. RAGHAVAN

1. Statement of the problem

Imagine that between two nodes there is a network of electrical connections, as for example in the following picture between nodes numbered 6 and 1:



Imagine further that between nodes 6 and 1 a voltage difference is forced, so that there is a current flowing from 6 to 1. Let us say that a current i_0 is flowing into node 6 and flowing out of node 1 as indicated by the arrows. Let us suppose that we are given information about the resistance of every edge. The problem to be solved is:

> How does the current i_0 distribute itself over the network? In other words, how to determine the current in each of the edges of the network (as a fraction of i_0)?

There is no loss of generality in assuming that the network contains no *loops* (that is, edges that start and end at the same node); that there is at most one edge between any two given nodes (multiple edges can be replaced by a single edge of appropriate resistance); and that the network is *connected* (if the "source" node, 6 in the picture above, is not connected to the "sink" node, 1 in the picture above, then of course no current can flow; nodes and edges that are not connected to the source and sink can safely be removed because no current flows in those edges).

We *orient* the edges (that is, put arrows on them) in an arbitrary fashion. Each edge has a node as *head* and another as *tail*. For example, 2 is the head and 3 the tail of the edge C in the above picture. Note that the orientation of an edge is without prejudice to the positive direction of actual

These are notes of a lecture given at Vellore Institute of Technology, Chennai, on 16th April 2015. They are an alternative exposition of the material in §8.2 of the book "Introduction to Linear Algebra" (South Asian Edition) by Gilbert Strang.

current flow: if it turns out that we chose the other direction, the algebraic value of the current in that edge will come out negative.

2. Comments on the solution

We will solve the above problem using linear algebraic notions, or if you prefer *matrix analysis*. We will set up a linear system of equations (see (6)) which we will show is solvable. Moreover, although (6) admits multiple solutions, all of them lead to the same values of currents in the edges. See the theorem in §6 and the remarks around it.

While we will illustrate the steps of the solution in the specific example of the network pictured above, the solution itself is general and applies to any problem of the above nature. It will take us on an excursion through several basic linear algebraic concepts. As we shall see, linear algebra is a good language in which to express the physics involved.

3. The adjacency matrix M

We begin by encoding into a matrix M, called the *adjacency matrix*, information about the network, excluding for the moment the information about the resistances of the edges. Let V denote the set of *nodes* or *vertices* of the network; let $\mathscr{E}$ denote the set of *edges* of the network. In the example pictured above, $\mathscr{V} = \{1, 2, 3, 4, 5, 6\}$ and $\mathscr{E} = \{A, B, C, D, E, F, G, H, I\}$. Let $|\mathscr{V}|$ and $|\mathscr{E}|$ denote respectively the cardinalities of $\mathscr{V}$ and $\mathscr{E}$. The adjacency matrix M is of size $|\mathscr{E}| \times |\mathscr{V}|$: its rows are indexed by $\mathscr{E}$, its columns by $\mathscr{V}$. If e is an edge and v a vertex, we will denote by $M(e, v)$ the entry in row e and column v of M. Each row of M has exactly two non-zero entries: if e be the edge indexing a row, $M(e, \text{head}(e)) = -1$ and $M(e, \text{tail}(e)) = 1$. The matrix M of the example pictured above is:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{1}$$

The rows correspond respectively to the edges A, B, C, …, H, I; the columns respectively to the nodes 1, …, 6.

Let us the pose the following questions about M:

What is the rank of M? What is its kernel? Its range?

Convention. In the sequel we will come across matrices with rows (or columns) indexed by $\mathscr{V}$ or $\mathscr{E}$. It is understood that the elements of $\mathscr{V}$ and $\mathscr{E}$ are enumerated once and for all in an arbitrary order. This order remains fixed throughout.

4. The rank of the adjacency matrix M

To find the rank of M, we may of course try to row reduce M. Exploiting the special nature of M—namely that it is the adjacency matrix of a network—leads us quickly to the answer. Let us denote the rows of M by ρ_A, ρ_B, etc. Observe that $\rho_A - \rho_B + \rho_C = 0$: this is because if we follow the network along the edge A, then along B but in the direction opposite to the arrow, and finally along C, then we would have completed a *cycle* (starting and ending with 2). So we may remove one of the rows ρ_A, ρ_B, ρ_C from the matrix M and without changing its row space. Let us say we remove ρ_B from M and correspondingly the edge B from the network.

Proceeding thus, we choose one cycle every time in the remaining network, observe a relation among the rows corresponding to the edges in the cycle, and remove an edge of the cycle from the network and the corresponding row from M. We continue until there are no more cycles left in the network.

In the pictured example above, we could proceed as follows:

- remove B (cycle, A, $-B$, C)
- remove D (cycle D, $-E$, G)
- remove F (cycle C, $-E$, F)
- remove I (cycle G, $-H$, I)

The removed rows ρ_B, ρ_D, ρ_F, and ρ_I being linear combinations of the rows that remain, the modified M has the same row space as M. We claim that the remaining rows, namely ρ_A, ρ_C, ρ_E, ρ_G, and ρ_H are linearly independent. This can be seen as follows. Suppose that there is a linear dependence relation $c_A\rho_A + c_C\rho_C + c_E\rho_E + c_G\rho_G + c_H\rho_H = 0$. Then, because ρ_H is the only row (among the five remaining rows) with a non-zero entry in column 6, it follows that $c_H = 0$. Of the other four rows rows (namely ρ_A, ρ_C, ρ_E, ρ_G), since ρ_G is the only row with a non-zero entry in column 4, it follows that $c_G = 0$. Of the remaining three rows (namely ρ_A, ρ_C, ρ_E), since ρ_E is the only one with a non-entry in column 5, it follows that $c_E = 0$. Since ρ_C has a non-entry in column 3 but ρ_A does not, it follows that $c_C = 0$. And finally $c_A = 0$ since $\rho_A \neq 0$. And the claim is proved.

We have just proved that the rank of the adjacency matrix M of the network pictured above is 5. The method of proof works in general and gives:

$$\text{The adjacency matrix has rank } |V| - 1. \tag{2}$$

Indeed, $|V| - 1$ is the number of edges that remain in the network at the point when there are no cycles left. And the rows corresponding to these edges are linearly independent.

5. The kernel and range of the adjacency matrix M

We think of M as being a linear transformation from $\mathbb{R}^{\mathscr{V}}$ to $\mathbb{R}^{\mathscr{E}}$. Elements of $\mathbb{R}^{\mathscr{V}}$ are naturally thought of as voltage values at the nodes (recall that X^Y denotes functions from Y to X). We may also think of $\mathbb{R}^{\mathscr{V}}$ as consisting of real column matrices of size $|\mathscr{V}| \times 1$. The image under M of an element $\underline{v}$ of $\mathbb{R}^{\mathscr{V}}$ is given by usual matrix multiplication: $\underline{v} \mapsto M\underline{v}$. Observe that $M\underline{v}$ is a column matrix of size $|\mathscr{E}| \times 1$ and therefore naturally denotes an element of $\mathbb{R}^{\mathscr{E}}$.

More abstractly (but also more intrinsically), if an element $\underline{v}$ of $\mathbb{R}^{\mathscr{V}}$ is interpreted as a real valued function on $\mathscr{V}$, its image under M is the real valued function on $\mathscr{E}$ that maps each edge e to the voltage difference between its tail and head, namely, $\underline{v}(\text{tail}(e)) - \underline{v}(\text{head}(e))$.

It is clear from the above description that if $\underline{v}$ is a constant function (in other words, if as a $|\mathscr{V}| \times 1$ column matrix, the entries of $\underline{v}$ are all equal), then its image under M is zero. On the other hand, since the rank of M is $|\mathscr{V}| - 1$, it follows from the rank-nullity theorem that the kernel of M has dimension 1. Thus:

$$\text{The kernel of } M \text{ consists of constant functions on } \mathscr{V}. \tag{3}$$

As to the range of M, which is a subspace of $\mathbb{R}^{\mathscr{E}}$, its dimension equals $|\mathscr{V}| - 1$ (see (2)). We think of elements of $\mathbb{R}^{\mathscr{E}}$ as real valued functions on $\mathscr{E}$. Suppose that $\epsilon_1 e_1, \ldots, \epsilon_k e_k$ is a cycle in the network. This means the following:

> Each ϵ_i has value either 1 or -1. Each e_i is an edge in the network. Let us say that to travel along $\epsilon_i e_i$ means to travel along the edge e_i, in the same direction as its arrow if $\epsilon_i = 1$, and in the opposite direction of its arrow if $\epsilon_i = -1$. Then traveling along successively along $\epsilon_1 e_1, \ldots, \epsilon_k e_k$ completes a cycle in the network. For example, A, -B, -F, E is a cycle in the network pictured above.

We may think of a cycle $\epsilon_1 e_1, \ldots, \epsilon_k e_k$ as a functional on $\mathbb{R}^{\mathscr{E}}$, just by letting f in $\mathbb{R}^{\mathscr{E}}$ be mapped to $\epsilon_1 f(e_1) + \cdots + \epsilon_k e_k$. We leave it to the reader as an exercise to prove the following assertions:

- The subspace spanned by the cycles in the dual space of $\mathbb{R}^{\mathscr{E}}$ has dimension $|\mathscr{E}| - |\mathscr{V}| + 1$. (In fact, any sequence of cycles chosen as in the procedure above for determining the rank of M would be a basis for the subspace spanned by the cycles.)
- An element f of $\mathbb{R}^{\mathscr{E}}$ belongs to the range of M if and only if, for every cycle $\epsilon_1 e_1, \ldots, \epsilon_k e_k$ in the network, we have $\epsilon_1 f(e_1) + \cdots + \epsilon_k f(e_k) = 0$.

6. Ohm's law

As seen in §4, we may think of the adjacency matrix M as a linear transformation from $\mathbb{R}^{\mathscr{V}}$ to $\mathbb{R}^{\mathscr{E}}$. Indeed if $\underline{v}$ is an element of $\mathbb{R}^{\mathscr{V}}$, then $M\underline{v}$, thought of as a function on $\mathbb{R}^{\mathscr{E}}$, has value $\underline{v}(\text{tail}(e)) - \underline{v}(\text{head}(e))$ at edge e. If we think of $\underline{v}$ as giving the voltages at the nodes, then $M\underline{v}$ evaluated at an edge e is just the voltage difference between its end points. Thus letting $\mathfrak{c}_e$ denote the conductance of edge e—recall that conductance is the reciprocal of resistance—we obtain, by Ohm's law, $\mathfrak{c}_e \cdot (M\underline{v})(e) = i_e$, where i_e denotes the current in edge e. We may combine these expressions of Ohm's law for every edge into a single matrix equation:

$$CM\underline{v} = \underline{i} \tag{4}$$

where C is a $|\mathscr{E}| \times |\mathscr{E}|$ diagonal matrix, whose rows and columns are both indexed by elements of $\mathscr{E}$ and whose diagonal entry corresponding to an edge e is the conductance $\mathfrak{c}_e$ of that edge; and $\underline{i}$ is the column matrix of size $|\mathscr{E}| \times 1$ whose entry corresponding to an edge e is the value of the current i_e in that edge (along the direction of the arrow).

7. The transpose of M and Kirchoff's current law

The transpose M^t of M is a $|\mathscr{V}| \times |\mathscr{E}|$ matrix, with rows indexed by $\mathscr{V}$ and columns by $\mathscr{E}$. Thus it is naturally interpreted as a linear transformation from $\mathbb{R}^{\mathscr{E}}$ to $\mathbb{R}^{\mathscr{V}}$. If we denote by $\underline{i}$ an element of $\mathbb{R}^{\mathscr{E}}$ and think of it as giving the values of the currents in the edges, then $M^t\underline{i}$ has a nice physical interpretation, which we now want to work out.

The entries along a given row of M^t are either 0, 1, or -1. In fact, if the row is indexed by a node v, then entry in a column corresponding to an edge e is 1 if v is the tail of e, -1 if v is the head of e, and 0 otherwise. Thus $M^t\underline{i}$ evaluated at a vertex v gives the value of the current flowing out of it (along the network's edges). This value is of course $-i_0$ at the sink, i_0 at the source, and 0 elsewhere (by Kirchoff's current law). Assuming we have enumerated the nodes in such a way that the first node is the sink and the last is the source, we obtain the matrix equation:

$$M^t \underline{i} = \begin{pmatrix} -i_0 \\ 0 \\ \vdots \\ 0 \\ i_0 \end{pmatrix} \tag{5}$$

8. The main equation and the main theorem

Combining equations (4) and (5) we get

$$M^t C M \underline{v} = \begin{pmatrix} -i_0 \\ 0 \\ \vdots \\ 0 \\ i_0 \end{pmatrix} \tag{6}$$

The matrices M and C represent the data of the connections in the network and the information about resistances of the edges. These are known. Thus we may solve the system (6) of linear equations for $\underline{v}$. The answers will be in terms of i_0. Once $\underline{v}$ is known, it is a simple matter to find the matrix $\underline{i}$ of currents: we could just plug $\underline{v}$ into (4) to get $\underline{i}$.

The questions that confront us now are:

- Is such a system as (6) always solvable?
- What if there are multiple solutions?

Our physical intuition tells us that there ought to be a unique solution to our original problem stated in §1. How is this captured by (6)?

As the following theorem shows, all is well:

Theorem. *The system (6) always has a solution. Any two solutions differ by a column matrix all of whose entries are equal.*

Thus while $\underline{v}$ itself admits multiple solutions, the voltage value at a given vertex by itself has no physical significance. It is only the differences in the voltages across edges which matter. And these are uniquely determined. Thus (6) is in agreement with our physical intuition.

9. Proof of the Theorem

We claim that $M^t CM$ has the same rank as M. For this, it is enough, by the rank nullity theorem, to show that $\operatorname{Ker} M = \operatorname{Ker} M^t CM$. The containment $\subseteq$ is clear. Now suppose that $M^t CM\underline{v} = 0$. Then, multiplying on the left by $\underline{v}^t$, we get $\underline{v}^t M^t CM \underline{v} = 0$ or $(M\underline{v})^t C(M\underline{v}) = 0$. Putting $\underline{x} = M\underline{v}$, we get $\underline{x}^t C \underline{x} = 0$ or $\sum_{e\in\mathscr{E}} \mathfrak{c}_e x_e^2$. Since $\mathfrak{c}_e > 0$ for all e and $x_e^2 \geq 0$ for all x_e, it follows that this can happen only if $\underline{x} = 0$. Thus $M\underline{v} = 0$, which means $\underline{v} \in \operatorname{Ker} M$ and the claim is proved.

By (2) it follows that $M^t CM$ has rank $|\mathscr{V}| - 1$. Now let us imagine solving (6) by row reducing the augmented matrix $(M^t CM|\underline{b})$ where $\underline{b}$ stands for the right hand side in (6). Observe that the sum of the columns of M is zero, which means that the sum of the rows of M^t (and therefore also those of $M^t CM$) is zero. Let us replace the last row of $(M^t CM|\underline{b})$ by the sum of all the rows. Then the last row would be zero, since $\underline{b}$ also has the property that the sum of all its rows is zero.

Let us imagine continuing with our row reduction of $(M^tCM|\underline{b})$. Since the rank of M^tCM is $|\mathscr{V}|-1$, all the rows except the last will be non-zero for the $|\mathscr{V}|\times|\mathscr{V}|$ matrix on the left. It follows that the system (6) has a solution, which proves the first assertion of the theorem.

Any two solutions of the inhomogeneous system (6) differ by a solution of the corresponding homogeneous system $M^tCM = 0$. As proved in the claim above, the kernel of M^tCM is the same as that of M. As observed in (3), the kernel of M consists of column matrices all of whose entries are the same, and the second assertion of the theorem is proved. $\square$