1. Tutorial 1a (VT 1)

Use ideas about group actions to prove (or disprove):

(1) If H and K are finite index subgroups of a group G, then so is $H \cap K$.

(2) For finite subgroups H and K of a group G, we have $|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$.

(3) Let H be a subgroup of a group G which acts on a set X. The following are equivalent:
- H acts transitively on X.
- G acts transitively on X and $G = HG_x$ for every x in X.
- G acts transitively on X and $G = HG_x$ for some x in X.

(4) There is no transitive action of the alternating group A_n (with $n \geq 5$) on a non-singleton set with less than n elements. (Note that the natural action of A_n on the set $\{1, 2, \ldots, n\}$ is transitive for $n \geq 3$.) (Hint: Given a group homomorphism $A_n \to S_m$ with $m < n$, this cannot be injective for $2 < n$, which means the kernel is a non-trivial normal subgroup.)

(5) Which of the following occur as the class equation of a group of order 10?:
- $10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$
- $10 = 1 + 2 + 3 + 4$
- $10 = 1 + 1 + 1 + 2 + 5$

2. TUTORIAL 1B (SV 1)

(1) By a *linear character of a group* G we mean a group homomorphism from $G \to \mathbb{C}^\star$. Every linear character of a group G factors through $G/[G,G]$, where $[G,G]$ is the commutator subgroup. A perfect group (that is, one whose commutator subgroup is itself, e.g., a non-abelian simple group) does not admit non-trivial linear characters.

(2) Let A be a finite abelian group and let $\rho : A \to GL_{\mathbb{C}}(V)$ be a finite dimensional complex representation. Show that there exists a basis of V each element of which is a common eigenvector for the action of all elements of A. Show that this is not true if we drop the hypothesis that A is abelian (in fact, for the symmetric group on three letters); show also that the statement is false for real representations in place of complex representations (in fact, even for the cyclic group of order three).

3. TUTORIAL 2A (VT 2)

(1) Let X be a finite set on which a group G acts transitively, and N be a normal subgroup of G. Then all N-orbits of X have the same cardinality.

(2) Let N be a normal subgroup of a group G that acts on a set X. Then there is a natural action of the quotient group G/N on $X^N := \{x \in X | nx = x \ \forall\ n \in N\}$.

(3) Any normal subgroup of a finite p-group intersects the centre of the group non-trivially.

(4) Let X be a finite set on which there is a transitive action of a group G. If Y is any finite G-set such that there is at least one G-map from Y to X, then the cardinality of X divides that of Y.

(5) Let Q_8 denote the group $\{\pm 1, \pm i, \pm j, \pm k\}$ (where i, j, k are quaternions in the standard notation). By Cayley's theorem, we have an embedding of Q_8 into the symmetric group of S_8. Show that Q_8 cannot be embedded into the symmetric group S_n for any $n < 8$. (Hint: Observe that $\{\pm 1\}$ is a subgroup of Q_8 that is contained in every non-trivial subgroup of Q_8. So if Q_8 has an action on any set of less than 8 elements, then this subgroup stabilises every point.)

(6) Suppose that the centre of group has finite index. Then the cardinality of any conjugacy class divides that finite index.

(7) A finite group has exactly two conjugacy classes. What can you say about it?

4. TUTORIAL 2B (SV 2)

(1) Every irreducible representation of a finite p-group over a field of characteristic p is one dimensional.

5. Tutorial 3a (VT 3)

(1) A normal p-subgroup of a finite group is contained in every Sylow p-subgroup.
(2) Consider the left action of a group (not necessarily finite) on itself and the induced action on the power set of the group. Observe that any subset S of the group is the union of (some) right cosets of its stabiliser. [Note: this observation is relevant to the proof of Sylow's existence theorem that was discussed in the lecture.]
(3) Given a non-singleton conjugacy class of a finite group G, there exists an element that does not commute with any element in the conjugacy class.
(4) If H is a finite index subgroup of a group G, then G is not the union of the conjugates of H.
(5) This shows that we cannot drop the hypothesis about H being of finite index in the previous item. Let $G = GL_n(\mathbb{C})$ (for some integer $n \geq 2$), and H be the subgroup of G consisting of invertible upper triangular matrices. Show that G is a union of conjugates of H.
(6) This gives an alternative proof of the existence of Sylow p-subgroups.
   (a) Show from first principles that, for a group G and a subgroup H, if G has a Sylow p-subgroup (for some prime p), then so does H. In fact, show more precisely the following: given a Sylow p-subgroup Q of G, the intersection of H with some conjugate of Q is a Sylow p-subgroup of H.
   (b) Use item (a) to give an alternative proof of the existence of Sylow p-subgroups. (Hint: Given a finite group G of order n, we have $G \hookrightarrow S_n$ by Cayley, where S_n denotes the symmetric group on n letters. Given a prime p, we can embed S_n further in $GL_n(\mathbb{F}_p)$. Now $GL_n(\mathbb{F}_p)$ has a Sylow p-subgroup, e.g., the subgroup consisting of unipotent upper triangular matrices.)

6. Tutorial 3b (SV 3)

(1) Let $\chi : A \to \mathbb{C}^*$ be a non-trivial character of a finite group A. Show that $\sum_{a\in A} \chi(a) = 0$. (Hint: Choose b in A such that $\chi(b) \neq 1$. Then $\sum_{a\in A} \chi(a) = \sum_{a\in A} \chi(ab) = \sum_{a\in A} \chi(a)\chi(b) = \left(\sum_{a\in A} \chi(a)\right)\chi(b)$, and so $\sum_{a\in A} \chi(a) = 0$.)

(2) Let A be any group, not necessarily finite, and let F be a field. Write $F[A]$ for the F-vector space of all functions from A to F. Recall that $\mathrm{Hom}(A, F^\times)$ denotes the group of all homomorphisms from A to the multiplicative group $F^\times$ of nonzero elements of F. Note that elements of $\mathrm{Hom}(A, F^\times)$ are functions on A valued in F, and hence $\mathrm{Hom}(A, F^\times) \subset F[A]$. Show: $\mathrm{Hom}(A, F^\times)$ is a linearly independent subset of $F[A]$.

Hint: Suppose not. Write a relation of linear independence of the form $c_1\chi_1 + \cdots + c_n\chi_n = 0$, where $c_i \neq 0$ for all i, where the χ_i are distinct, and where n is the smallest possible for such a relation. This means $c_1\chi_1(a) + \cdots + c_n\chi_n(a) = 0$ for all $a \in A$, and hence also for a replaced by ab with $a, b \in A$. Since $\chi_1 \neq \chi_2$, take b such that $\chi_1(b) \neq \chi_2(b)$. Now you have two equations, manipulate them to eliminate χ_1 and get a nontrivial relation of linear dependence with at most $n - 1$ terms, to obtain a contradiction.

Note: The fact asserted in this question is the well-known 'linear independence of characters'.

(3) Let A be a finite abelian group.

(a) Suppose $a \in A$. Show:

$$\sum_{\chi\in \mathrm{Hom}(A,\mathbb{C}^\times)} \chi(a) = \begin{cases} |\mathrm{Hom}(A, \mathbb{C}^\times)|, & \text{if } a = e \text{, where } e \text{ is the identity element of } A\text{, and} \\ 0, & \text{if } a \neq e. \end{cases}$$

Hint: For the case $a \neq e$, use the structure theorem for finite abelian groups to get that there exists $\chi_0 \in \mathrm{Hom}(A, \mathbb{C}^\times)$ with $\chi_0(a) \neq 1$. Show that the left-hand side of the above equation remains unchanged on multiplying with $\chi_0(a)$. Why does that force the left-hand side to be zero?

(b) Suppose $a_0 \in A$. Show that the function on A given by:

$$\delta_{a_0} := \frac{1}{|\mathrm{Hom}(A, \mathbb{C}^\times)|} \sum_{\chi\in \mathrm{Hom}(A,\mathbb{C}^\times)} \chi(a_0)^{-1}\chi,$$

namely the function defined by

$$\delta_{a_0}(a) = \frac{1}{|\mathrm{Hom}(A, \mathbb{C}^\times)|} \sum_{\chi\in \mathrm{Hom}(A,\mathbb{C}^\times)} \chi(a_0)^{-1}\chi(a),$$

is given by:

$$\delta_{a_0}(a) = \begin{cases} 1, & \text{if } a = a_0\text{, and} \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Use part (a).

(c) Conclude from (b) that $\mathrm{Hom}(A, \mathbb{C}^\times) \subset \mathbb{C}[A]$ is a spanning set of the $\mathbb{C}$-vector space $\mathbb{C}[A]$.

(d) Conclude from problems 1 and 2(c) that $\mathrm{Hom}(A, \mathbb{C}^\times) \subset \mathbb{C}[A]$ is a basis of the $\mathbb{C}$-vector space $\mathbb{C}[A]$ (and thence also that $|A| = |\mathrm{Hom}(A, \mathbb{C}^\times)|$).

Note: We discussed the inner product on $\mathbb{C}[A]$:

$$\langle f_1, f_2\rangle = \frac{1}{|A|} \sum_{a\in A} f_1(a)\overline{f_2(a)},$$

and the fact that $\mathrm{Hom}(A, \mathbb{C}^\times)$ is an orthonormal set for this inner product. Thus, from the above problems it follows that $\mathrm{Hom}(A, \mathbb{C}^\times)$ is an orthonormal basis for this inner product. Moreover, each of these elements spans an irreducible subrepresentation of $\mathbb{C}[A]$ for the left (or right) regular action.

(4) Suppose $\rho : G \to GL_n(\mathbb{C}) = GL_{\mathbb{C}}(\mathbb{C}^n)$ be a complex representation of a group G. Thus, for every $g \in G$, $\rho(g)$ is an $n \times n$ matrix. Let $r_{ij}(g)$ denote the (i, j)-th entry of $\rho(g)$. Each r_{ij} is then a function from G to $\mathbb{C}$, $g \mapsto r_{ij}(g)$; in other words, $r_{ij} \in \mathbb{C}[G]$.

Show using matrix multiplication and the fact that ρ is a homomorphism: if ρ_{lreg} denotes the left regular representation of G on $\mathbb{C}[G]$, then for all $h \in G$:

$$\rho_{\mathrm{lreg}}(h)(r_{ij}) = \sum_{k=1}^{n} r_{ik}(h^{-1}) \cdot r_{kj}.$$

Conclude that the span of all the r_{ij} is a G-invariant subspace of $(\mathbb{C}[G], \rho_{\mathrm{lreg}})$.
Prove the analogous assertions for the right regular representation ρ_{rreg} of G on $\mathbb{C}[G]$ and the regular representation ρ_{reg} of $G \times G$ on $\mathbb{C}[G]$.

(5) Let (V, ρ) be a representation of a finite group G, over a field F.
   (a) Let $W \subset V$ be a finite dimensional vector subspace, and set:

   $$W' = \text{the } F\text{-linear span of } \{\pi(g)w \mid g \in G, w \in W\}.$$

   Show that W' is a G-invariant subspace of V that contains W, of dimension at most $|G| \cdot \dim_F W$.
   (b) If W'' is any G-invariant subspace of V that contains W, show that W'' contains W'.
   (c) Using (a) or otherwise, show that every irreducible representation of a finite group is finite dimensional.

   Note: (b) tells us that W' can be thought of as 'the' (and not just 'a') smallest G-invariant subspace of V containing W. W' is said to be the G-invariant subspace of V generated by W.

(6) This gives a representation theoretic proof of the Cauchy-Frobenius-Burnside orbit counting lemma:
   (a) Suppose that V is a finite dimensional complex representation of a finite group G defined by $\rho : G \to GL(V)$. Let V^G be the subspace consisting of the points of V that are fixed by G. Show that the following endomorphism of V defines a G-projection onto the subspace V^G:

   $$\frac{1}{|G|} \sum_{g \in G} \rho(g)$$

   (b) The trace of any projection to a subspace equals the dimension of the subspace.
   (c) Let X be a finite set on which a group acts. Consider the the free complex vector space V on X as a linear representation of G. Then the dimension of V^G equals the number of G-orbits in X. [Hint: First assume that the action of X is transitive and observe that V^G is one dimensional in this case.]
   (d) Prove the Cauchy-Frobenius-Burnside orbit counting lemma using the three items above.

(7) If every irreducible complex representation of a finite group is one dimensional, then the group is abelian.

7. TUTORIAL 4A (VT 4)

(1) If there is exactly one Sylow p-subgroup of a finite group for every prime p, then the group is a direct product of its Sylow p-subgroups.

(2) Let G act transitively on X and Y. The natural action of G on $X \times Y$ is transitive if and only if $G_x G_y = G$ for some $x \in X$ and $y \in Y$ (equivalently for every $x \in X$ and $y \in Y$).

(3) Let G be a finite group, S a subgroup, and P a Sylow p-subgroup of S (for some prime p). Put $N(P;S) := \{g \in G \mid gPg^{-1} \subseteq S\}$. Observe that $N(P;S)$ contains the identity element, is stable under left multiplication by $N_G(S)$ and under right multiplication by $N_G(P)$. In particular, it contains both $N_G(S)$ and $N_G(P)$. Show that $N(P;S) \subseteq SN_G(P)$, and thus in particular that $SN_G(P) = G$ if S is normal in G. Other consequences are derived in the next few items. (Hint: Suppose that $g^{-1}Pg \subseteq S$. Then $g^{-1}Pg$ is a Sylow-p subgroup of S, and, by Sylow's second theorem, there exists $s \in S$ such that $s^{-1}Ps = gPg^{-1}$, which means sg belongs to $N_G(P)$, and so $g = s^{-1}(sg) \in SN_G(P)$.)

(4) Let G be a finite group and S a subgroup that contains the normaliser $N_G(P)$ of a Sylow-p subgroup P of S (not necessarily of G). Then P is not contained in any other conjugate of S. More strongly, if $P \subseteq gSg^{-1}$, then $g \in S$. Observe that this implies the following: S is its own normaliser. (Hint: Suppose that $P \subseteq gSg^{-1}$. Then $g^{-1}Pg \subseteq S$, and so, by item (3), $g^{-1} \in N(P;S) \subseteq SN_G(P)$. But by the hypothesis that $N_G(P) \subseteq S$, it follows that $g^{-1} \in N(P:S) \subseteq S$.)

(5) As already seen in item (4), a subgroup of a finite group that contains the normaliser of one of its (the subgroup's, not necessarily the whole group's) Sylow p-subgroups is its own normaliser. Observe that this is equivalent to saying: if a subgroup of a finite group is normal and contains the normaliser of one of its (the subgroup's) Sylow-p subgroups, then it is the whole group. (Hint: Let P be a Sylow-p subgroup of a subgroup S of a finite group G, and suppose that $S \supseteq N_G(P)$. If $N_G(S) \supsetneq S$, then the second statement is violated if we take $N_G(S)$ to be the whole group. Conversely, if the second statement is violated, that is, if S is normal and proper, then the first statement is violated, since $N_G(S) = G \supsetneq S$.)

(6) Let G be a finite group, S a subgroup, and P a Sylow-p subgroup of S (not necessarily of G). Suppose that $N_G(P) \subseteq S$. Then, for any subgroup T of G containing S, we have $[T : S] \equiv 1 \bmod P$. (Hint: May assume $T = G$ without loss of generality. Consider the action of P on the set $X = G/S$. We have $|X| = [G : S]$. (Although we don't need this fact, observe that X may be identified with the conjugates of S since S is its own normaliser by item (5).) Since $|X| = |X^P| \bmod p$, it is enough to show that P does not fix any coset gS of S other than S. Let $PgS = gS$. Then $Pg \subseteq gS$, and so $P \subseteq gSg^{-1}$. By item (4) we have $g \in S$, and so $gS = S$.)

7.1. **Some standard applications of Sylow's theorems.**

(1) Prove that there are no simple groups of order 12, 21, 56, 72, or 96.

(2) Any Sylow-11 subgroup of a group of order 231 is contained in the centre.

8. Tutorial 6a (VT 5)

(1) Let V and W be vector spaces, V finite dimensional. The map $V^\star \otimes W \to \operatorname{Hom}(V, W)$ given by the following description is an isomorphsim: the image of $\varphi \otimes w$ acting on v is $\varphi(v)w$. Describe the inverse of this map.

(2) A conceptual description of trace: Let V be a finite dimensional vector space. We have the natural isomorphism $\operatorname{End} V \simeq V^\star \otimes V$. We also have the linear map $V^\star \otimes V \to \mathbb{F}$ given by $\psi \otimes v \mapsto \psi(v)$, where $\mathbb{F}$ is the underlying field. The composition $\operatorname{End} V \to \mathbb{F}$ is the trace map.

9. Tutorial 6b (KNR1)

(1) Let V and W be finite dimensional complex representations of a finite group G. Let χ_V and χ_W be their respective characters. Then
   (a) $\chi_{V\oplus W} = \chi_V + \chi_W$.
   (b) $\chi_{V^\star} = \overline{\chi_V}$. (Hint: If M is the matrix of g_V with respect to a basis of V, then the matrix is $M^\star$ of $g_{V^\star}$ with respect to the dual basis of $V^\star$.)
   (c) $\chi_{V\otimes W} = \chi_V\chi_W$. (Hint: If $M = (m_{ii'})$ be the $m \times m$ matrix of g_V with respect to a basis $\{v_i\}$ of V and $N = (n_{jj'})$ the $n \times n$ matrix of g_W with respect to a basis $\{w_j\}$ of W, then $M \otimes N$, defined to be the $mn \times mn$ matrix whose entry in position $ij, i'j'$ is $m_{ii'}n_{jj'}$, is the matrix of $g_{V\otimes W}$ with respect to the basis $\{v_i \otimes w_j\}$ of $V \otimes W$.)
   (d) $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$. (Hint: $\mathrm{Hom}(V, W) \simeq V^\star \otimes W$ is a G-isomorphism; now use the previous two items.)

(2) Let V and W be linear representations of a group G. Then $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$. (Proof: $g\varphi g^{-1} = \varphi \Leftrightarrow g\varphi = \varphi g$.)

(3) Let $G = \{\pm 1\}$ be the subgroup of order two of the group of non-zero real numbers (under multiplication). Consider the action of G on $\mathbb{R}$ by left multiplication, and the "induced" (or "derived") action on real functions on $\mathbb{R}$: $gf(x) := f(g^{-1}x)$. A function $f(x)$ is G-invariant if and only if it is even. The "average" of a general function $f(x)$ is $(f(x) + f(-x))/2$, which is always even.

10. Tutorial 6b (KNR 1) Continued

(1) Recall the following from an early lecture by SV. Let $\rho : G \to GL(V)$ be a finite dimensional complex representation of a finite group G. Then, for any element g of the group G, the linear transformation $\rho(g)$ is diagonalizable. (Hint: If n is such that $g^n = 1$, then $\rho(g)$ satisfies the equation $X^n - 1 = 0$, which means that its minimal polynomial divides $X^n - 1$, which has distinct roots.) In fact, $\rho(g)$ can be represented by a diagonal matrix each of whose diagonal entry is a complex root of unity.

Caveat: While each $\rho(g)$ is by itself diagonalizable, this does not mean that we can necessarily simultaneously diagonalize all of them. In fact, $\rho(g)$ are simultaneously diagonalizable if and only if they all commute with one another.

Prove the following using the above ideas:

(a) Any irreducible complex representation of a finite abelian group is one dimensional.

(b) Let χ_V be the character of a finite dimensional complex representation of a finite group G. Then:

(i) $|\chi_V(g)| \leq \dim_{\mathbb{C}} V$ for every $g \in G$, and equality holds if and only if g acts on V as multiplication by a scalar (complex number).

(ii) $\chi_V(g)$ is an integer if it is rational. (Hint: Diagonalize the linear transformation g_V. It is then clear that $\chi(g)$ is a sum of complex roots of unity. As such, it is an algebraic integer. But only the integers among rationals are algebraic integers.)

(c) The centre of a finite group G consists of those elements g for which $|\chi_V(g)| = \chi_V(1) = \dim_{\mathbb{C}} V$ for every finite dimensional complex irreducible representation V of G.

(2) The purpose of this exercise is to illustrate that ideas about finite group actions on sets can be applied to representations. Let G be a p-group and V a finite dimensional linear representation of G over a finite field of characteristic p. If V is non-zero, then there exists a non-zero vector v in V that is left invariant by G, that is, $gv = v$ for all g in G. (Hint: $|V| = |V^G| \bmod p$.)

11. TUTORIAL 7A (GT 1)

In what follows, R denotes a ring with identity, not necessarily commutative.

(1) (Examples of division rings) Quaternions $\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}; \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}$ form a division ring. We could take the coefficients to range over some subfield of $\mathbb{R}$ (e.g., $\mathbb{Q}$) rather than over all of $\mathbb{R}$, to get other division rings.

(2) Suppose that every non-zero element of R has a left (multiplicative) inverse. Then every non-zero element of R has a two-sided inverse. (Hint: Let b be a left inverse of a, and c a left inverse of b. Then $c = c \cdot 1 = c(ba) = (cb)a = 1 \cdot a = a$. Thus a is a left inverse of b, which is the same thing as saying b is a right inverse of a.)

(3) The matrix ring $M_n(D)$ (n is a positive integer) over a division ring D is a simple ring.

(4) Let R_ℓ denote R considered as a left module over itself. Show that R_ℓ is a simple module if and only if R is a division ring.

(5) Schur's Lemma:
- A non-zero R-module homomorphism $M \to N$ between simple R-modules is an isomorphism. In particular, $\mathrm{End}_R(M)$ is a division ring for M a simple R-module.
- Suppose that a finite dimensional $\mathbb{C}$-algebra A is a division ring. Show that A must be $\mathbb{C}$ itself (more precisely, the ring homomorphism $\mathbb{C} \to A$ that defines A as a $\mathbb{C}$-algebra is an isomorphism onto A).
- Deduce the following: if R is a $\mathbb{C}$-algebra and M a finite dimensional simple R-module, then $\mathrm{End}_R(M)$ consists only of homotheties.

(6) Every two sided ideal of $M_n(R)$ has the form $M_n(I)$ for some unique two sided ideal I of R.

(7) For D a division ring, $M_n(D)$ is a direct sum of n minimal left ideals.

(8) Any simple module is cyclic. Is the coverse true?

(9) Determine all simple $\mathbb{Z}$-modules.

(10) Any simple module is indecomposable. How about the converse?

12. Tutorial 7B (KNR 2)

(1) Given a group homomorphism $\rho : G \to GL_n(\mathbb{R})$ of a finite group G, show that there exists an element P in $GL_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is, for every g in G, an orthogonal $n \times n$ matrix.

13. Tutorial 8A (GT 2)

(1) Is the zero module simple? Is it semisimple?
(2) Every submodule of a semisimple module has a complement. Is it (the complement) unique?
(3) Submodules and quotient modules of semisimple modules are semisimple.
(4) What are all the semisimple $\mathbb{Z}$-modules?
(5) Write down all composition series for the cyclic abelian group $\mathbb{Z}/6\mathbb{Z}$.
(6) Write down all composition series for the cyclic abelian group $\mathbb{Z}/n\mathbb{Z}$.
(7) Any simple $M_n(D)$-module (where D is a division ring) is isomorphic to D^n.

14. Tutorial 8A (GT 2) Continued

Let R be an (assoicative) algebra over a field k, and let M be a finite dimensional (left) R-module.

(1) Example of a simple module that does not occur as a submodule of the ring (as a left module over itself). Recall that every simple module occurs as the quotient of the ring (as a left module over itself). Let A be the subalgebra of $M_2(\mathbb{C})$ consisting of the upper triangular matrices. Let I be the left ideal of A consisting of those matrices whose second row is identically zero. Show that A/I is a simple A-module and that there is no submodule of A that is isomorphic to A/I.

(2) A sequence $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_{k-1} \subsetneq M_k = M$ of submodules such that each quotient M_j/M_{j-1} is simple (for $1 \leq j \leq k$) is said to be a *composition series*. Such a sequence always exists (because of the finite dimensionality of M). We call k the *length* of the composition series and the collection (with multiplicity) of (the isomorphism classes of) simple modules $\{M_j/M_{j-1} \mid 1 \leq j \leq k\}$ the *composition factors*.

(3) (Jordan Hölder Theorem) The length of any composition series and the composition factors in it depend only upon M and not on the particular composition series. (Hint: Proceed by induction on the dimension of M.)

15. Tutorial 8B (KNR 3)

(1) Let G be a finite group. Let V_1, V_2, $\dots$ be a complete list of representatives of isomorphism classes of irreducible representations, and let χ_1, χ_2, $\dots$ be their respective characters. Let V be a finite dimensional complex representation. Suppose that a decomposition of V as a direct sum of irreducible representations is given. Let m_1, m_2, $\dots$ be the numbers of the summands that are isomorphic respectively to V_1, V_2, $\dots$. Use the orthonormality of χ_1, χ_2, $\dots$ to show the following:

(a) the numbers m_1, m_2, $\dots$ depend only upon V and not upon the particular decomposition as a direct sum of irreducibles. In fact, $m_j = \langle \chi_V, \chi_j \rangle$, and so is determined by χ_V.

(b) Characters determine representations: that is, two representations that have the same character are isomorphic. (Hint: This follows from the previous item.)

(c) $\langle \chi_V, \chi_V \rangle = m_1^2 + m_2^2 + \cdots$. In particular, V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

(d) The number of isomorphism classes of irreducible representations is at most the number of conjugacy classes. (In particular, the list V_1, V_2, $\dots$ is finite.) **Note:** In fact, equality holds since χ_1, χ_2, $\dots$ span the space of class functions on G as we have further shown.

(2) Fix notation as in the previous item. Deduce from Schur's lemma that, for a complex finite dimensional representation V' of G, $\boxed{\dim \mathrm{Hom}_G(V, V') = m_1 m_1' + m_2 m_2' + \cdots}$, where $m_j' := \langle \chi_{V'}, \chi_j \rangle$.

(3) (Decomposition of the left regular representation) Fix notation as in item (1) above. Let Λ denote the left regular representation of G and χ_Λ its character.

(a) $\chi_\Lambda(g) = 0$ unless $g = 1$, and $\chi_\Lambda(1) = |G|$.

(b) $\langle \chi_\Lambda, \chi_j \rangle = \dim V_j$. Thus each irreducible representation V_j occurs in Λ with multiplicity $\dim V_j$.

(4) Let G be a finite group and V a finite dimensional complex representation. Then $\chi_V(1) = \dim V$, where χ_V is the character of V.

(5) Let G be a finite group[1] and $\mathbb{C}G$ the group ring of G with complex coefficients. The purpose of this exercise is to show that a complex representation of G is nothing more and nothing less than a $\mathbb{C}G$-module V. Thus ideas and constructs from ring and module theory can be brought to bear upon questions about representations.

- Recall that a complex vector space V is said to be a $\mathbb{C}G$-*module* if there exists a $\mathbb{C}$-algebra homomorphism $\rho : \mathbb{C}G \to \mathrm{End}_{\mathbb{C}}(V)$. Since elements of G (identified with their canonical images in $\mathbb{C}G$) are units in $\mathbb{C}G$, their images $\rho(g)$, $g \in G$, are units in $\mathrm{End}_{\mathbb{C}}(V)$. Thus, restricting ρ to G, we get a map $G \to GL(V)$. This is moreover a group homomorphism.
- Conversely, suppose we are given a linear representation $\rho : G \to GL(V)$ of G on a complex vector space V. Then, by a linear extension, we get a homomorphism $\mathbb{C}G \to \mathrm{End}_{\mathbb{C}}(V)$ of vector spaces. This is moreover an algebra homomorphism.
- The space $\mathrm{Hom}_G(V, W)$ of G-homomorphisms between two complex representation spaces V and W of G is the same as the space $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ of $\mathbb{C}G$-module homomorphisms when we think of V and W as $\mathbb{C}G$-modules.

(6) Let G be a finite group and c the number of its conjugacy classes. Let C_1, $\dots$, C_c be all the conjugacy classes in G. The elements χ_i, $1 \le i \le c$, where $\chi_i = \sum_{g \in C_i} g$, form a basis for the centre of $\mathbb{C}G$. (Hint: $h\chi_i h^{-1} = \sum_{g \in C_i} hgh^{-1} = \sum_{g \in C_i} g$.) Thus the centre of the group ring $\mathbb{C}G$ has dimension c as a complex vector space. An element $\sum_{g \in G} f(g)g$ in $\mathbb{C}G$ belongs to the centre if and only if f is a class function on G.

[1] The assumption here of finiteness of G is not really necessary.

16. Tutorial 9A (GT 3)

(1) Let R_ℓ denote R considered as a left module over itself. Show that $\operatorname{End}_R R_\ell$ is isomorphic as a ring to R^{opp}, where R^{opp} is the opposite ring of R.
(2) $M_n(R)^{\text{opp}} = M_n(R^{\text{opp}})$, for any ring R.
(3) A semisimple ring (not necessarily a finite dimensional algebra over a field) admits only finitely many isomorphism classes of simple modules.

17. TUTORIAL 9B (KNR 4)

(1) Let G be a group and X a set with a G-action. We denote by $\mathbb{C}X$ the free $\mathbb{C}$-vector space with X as basis and by $\mathbb{C}[X]$ the $\mathbb{C}$-vector space of functions on X with values in $\mathbb{C}$. (Recall that both these are representations of G: the action of G on $\mathbb{C}X$ is obtained by extending linearly the action on X; the action on $\mathbb{C}[X]$ is given by the following: $(fg)(x) = f(g^{-1}x)$ for f in $\mathbb{C}[X]$, g in G, and x in X.) For x in X, let δ_x denote the function on X that takes value 1 on x and 0 elsewhere.

Assume that X is finite. Verify the following:

(a) The map $x \mapsto \delta_x$ defines a G-linear isomorphism between the G-spaces $\mathbb{C}X$ and $\mathbb{C}[X]$. (What happens if the finiteness assumption on X is removed?)

(b) For $g \in G$, the value $\chi_{\mathbb{C}X}(g)$ at g of the character $\chi_{\mathbb{C}X}$ of $\mathbb{C}X$ is $|X^g|$.

(c) $\mathbb{C}[X]_0 := \{f \in K[X] \mid \sum_{x\in X} f(x) = 0\}$ is a G-invariant subspace of $\mathbb{C}[X]$. Evidently, constant functions span a one dimensional G-invaraint subspace of $\mathbb{C}[X]$. These two subspaces are complementary.

(2) Let G be a finite group and V, V' finite dimensional complex representations of G.

(a) V is called *self dual* if it is isomorphic (as a representation) to its dual $V^\star$. Observe that V is self dual if and only if its character takes only real values.

(b) $\dim V^G = \dim (V^\star)^G$

(c) $\dim \mathbf{Hom}_G(V, V') = \langle \chi_V, \chi_V' \rangle$

18. Tutorial 10A (GT4)

(1) Let G be a finite group and consider the left regular representation on $\mathbb{C}G$ of G. The elements of G form a basis for $\mathbb{C}G$. The action of each element of G with respect to this basis is represented by a permutation matrix. Write these matrices down explicitly when G is the cyclic group of three elements and when G is the Klein four group.

(2) Let G be the subgroup of $\mathfrak{S}_5$ generated by a 5-cycle. Let $X = \{1, 2, 3, 4, 5\}$ and consider the restriction to G of the defining action of $\mathfrak{S}_5$ on $\mathbb{C}X$. Prove that $\mathbb{C}X$ is isomorphic as a G-representation to the left regular representation of G.

(3) Let G be the subgroup of the symmetric group $\mathfrak{S}_4$ generated by (12) and (34). Let $X = \{1, 2, 3, 4\}$ and consider the restriction to G of the defining action of $\mathfrak{S}_4$. Is $\mathbb{C}X$ isomorphic as a G-representation to the left regular representation of G?

19. TUTORIAL 10B (KNR 4)

(1) For X a finite G-set, $\dim(\mathbb{C}X^G)$ equals the number of G-orbits in X.

(2) Let G be a group and X, Y be G-sets. Consider the following "integration along the fiber" map:

$$T:\ \mathbb{C}[X\times Y]\to \mathbf{Hom}_{\mathbb{C}}(\mathbb{C}X,\mathbb{C}[Y]),\quad k(x,y)\mapsto T_k,\text{ where } T_k(\sum_{x\in X} f(x)x)(y):=\sum_{x\in X} f(x)k(x,y)$$

The summation makes sense since the function $f(x)$ on X has finite support. Show that:

(a) T is linear map of vector spaces.
(b) T is one-to-one. (Hint: $k(x,y)$ can be recovered from its image T_k. Indeed $k(x,y)=T_k(x)(y)$.)
(c) T is onto. (Hint: For x_0 in X and y_0 in Y, let F_{x_0,y_0} be the linear function from $\mathbb{C}X$ to $\mathbb{C}[Y]$ that maps x_0 to δ_{y_0} and remaining x to 0. It is enough to show that F_{x_0,y_0} belongs to the image of T, for any choice of x_0 and y_0. Now check that T maps δ_{x_0,y_0} to F_{x_0,y_0}.)
(d) Check that T commutes with the action of G on either side.
(e) Conclude that $\mathbb{C}[X\times Y]^G\simeq \mathbf{Hom}_G(\mathbb{C}X,\mathbb{C}[Y])$.
(f) (Intertwining Number Lemma) Now suppose that X and Y are finite. Conclude that $\dim \mathbf{Hom}_G(\mathbb{C}X,\mathbb{C}[Y])$ equals the number of G-orbits in $X\times Y$. (Hint: Use the previous item of this exercise and the previous exercise.)

(3) Let n be a natural number. For i a non-negative integer, $i\le n$, let X_i denote the set of subsets of cardinality i of $[n]=\{1,2,\ldots,n\}$. The defining action of the symmetric group $\mathfrak{S}_n$ on $[n]$ induces an action on X_i. Let k and ℓ denote non-negative integers $\le n$.

(a) The action of $\mathfrak{S}_n$ on X_k is transitive.
(b) $X_k\simeq X_{n-k}$ as $\mathfrak{S}_n$-sets. (Hint: $S\leftrightarrow [n]\setminus S$.)
(c) Observe that two elements (S,T) and (S',T') in $X_k\times X_\ell$ are in the same $\mathfrak{S}_n$-orbit (for the diagonal action on $X_k\times X_\ell$) if and only if $|S\cap T|=|S'\cap T'|$.
(d) Conclude from the previous item that, for $0\le k,l\le n/2$, the number of G-orbits in $X_k\times X_\ell$ equals $\min\{k,\ell\}+1$.
(e) Use the previous item and the intertwining number lemma (previous exercise) to conclude, for $0\le k,\ell\le n/2$, that $\dim \mathbf{Hom}_{\mathfrak{S}_n}(\mathbb{C}X_k,\mathbb{C}X_\ell)=\min\{k,\ell\}+1$. (Hint: $\mathbb{C}Y\simeq\mathbb{C}[Y]$ as G-representations, as was seen in an earlier tutorial problem.)

(4) (Alternative proof of the intertwining number lemma, using the orbit counting lemma) Let X and Y be finite sets with actions of a finite group G. Then

$$\begin{aligned}
\dim \mathbf{Hom}_G(\mathbb{C}X,\mathbb{C}[Y]) &= \langle \chi_{\mathbb{C}X},\chi_{\mathbb{C}[Y]}\rangle \quad (\text{since } \dim \mathbf{Hom}_G(V,V')=\langle\chi_V,\chi_{V'}\rangle, \text{ as seen earlier})\\
&= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\mathbb{C}X}(g)}\chi_{\mathbb{C}[Y]}(g) \quad (\text{definition of the inner product})\\
&= \frac{1}{|G|}\sum_{g\in G}\chi_{\mathbb{C}X}(g)\chi_{\mathbb{C}Y}(g) \quad (\chi_{\mathbb{C}X}(g) \text{ is an integer and } \mathbb{C}Y\simeq\mathbb{C}[Y])\\
&= \frac{1}{|G|}\sum_{g\in G}|X^g|\cdot|Y^g| \qquad (\chi_{\mathbb{C}X}(g)=|X^g|)\\
&= \frac{1}{|G|}\sum_{g\in G}|(X\times Y)^g| \quad (G \text{ acts diagonally on } X\times Y)\\
&= \text{number of } G\text{-orbits in } X\times Y \quad (\text{orbit counting lemma})
\end{aligned}$$

20. TUTORIAL 11A (GT 5)

(1) Prove that if a finite group admits a faithful irreducible representation, then its centre is cyclic.

(2) Let g be a generator of the cyclic group G of order 3. Verify that g mapping to the matrix $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ defines a complex representation of G. Decompose the representation into irreducibles.

(3) (Wedderburn decomposition) Let G be the Klein four group. Write the group ring $\mathbb{C}G$ as the direct product of four one dimensional subalgebras.

21. TUTORIAL 11B (KNR 6)

(1) Enumerate all SSYT of shape $5+3$ and type $3+3+2$. How many of them are there?

(2) Let $\lambda : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$ and $\mu : \mu_1 \geq \mu_2 \geq \ldots \mu_m \geq 1$ be two partitions of a positive integer n. Let X_λ (respectively X_μ) be the collection of set partitions of $[n] = \{1, 2, \ldots, n\}$ of type λ (respectively μ). Two elements $(\mathbb{S}, \mathbb{T})$ and $(\mathbb{S}', \mathbb{T}')$ of $X_\lambda \times X_\mu$ are in the same orbit of $\mathfrak{S}_n$ if and only if $|S_i \cap T_j| = |S'_i \cap T'_j|$ for every $1 \leq i \leq l$ and $1 \leq j \leq m$, where $\mathbb{S}$ denotes the set partition $S_1 \sqcup \cdots \sqcup S_l = [n]$ with $|S_i| = \lambda_i$ and $\mathbb{T} = T_1 \sqcup \ldots \sqcup T_m = [n]$ with $|T_j| = \mu_j$.

Denote by $\mathbb{M}_{\lambda\mu}$ the set of non-negative integer matrices of size $l \times m$ with row sums being $\lambda_1, \ldots, \lambda_l$ and column sums being $\mu_1, \ldots, \mu_m$. Denote by $M_{\lambda\mu}$ the cardinality of $\mathbb{M}_{\lambda\mu}$. From the intertwining number lemma and the observation above about the G-orbits in $X_\lambda \times X_\mu$, deduce that $M_{\lambda\mu} = \dim \mathbf{Hom}_G(\mathbb{C}X_\lambda, \mathbb{C}Y_\mu)$.

22. TUTORIAL 12A (GT 6)

Some suggested references:

- Karin Erdmann: Representations of Algebras
- Bruce Sagan: Representations of symmetric groups
- Gordon James and Martin Liebeck: Representations of finite groups

(1) For a finite group G, the number of one dimensional representations is equal to the order of G/G', where G' is the commutator subgroup of G. Given that the alternating group A_n is the commutator subgroup of $\mathfrak{S}_n$, it follows that $\mathfrak{S}_n$ has exactly two irreducible one dimensional representations. These are the trivial and sign representations.

(2) How many irreducible representations (up to isomorphism) does A_4 have? What are their dimensions? Find the Wedderburn decomposition as a product of matrix rings of $\mathbb{C}A_4$. (Hint: The commutator subgroup of A_4 is $\{\text{identity}, (12)(34), (13)(24), (14)(23)\}$. So there are three one dimensional representations.)

(3) Suppose that a finite group G has an abelian normal subgroup N. Then any irreducible representation of G has dimension bounded by $|G/N|$. Observe that this implies that the irreducible representations of dihedral groups D_n are at most two dimensional. Work out the Wedderburn decomposition of the dihedral group D_{10} of order 10.

(4) What is the Wedderburn decomposition of $\mathbb{C}S_3$?

(5) Let $A = k[X]/(f)$, where $f \in k[X]$ is the product $f_1 \cdots f_r$ of distinct irreducible monic polynomials $f_1, \ldots, f_r$. Then the Wedderburn decomposition of the semisimple algebra A is the isomorphism $A \simeq k[x]/(f_1) \times \cdots \times k[x]/(f_r)$ given by the Chinese Remainder Theorem.

(6) The Wedderburn decomposition of the group ring kC_n of the cyclic group C_n of order n with coefficients in a field k is given by the decomposition of the previous item taking f to be the polynomial $f(x) = x^n - 1$.

(7) The Wedderburn decomposition of $\mathbb{R}C_3$ is given by $\mathbb{R}C_3 \simeq \mathbb{R}[X]/(x-1) \times \mathbb{R}[x]/(x^2+x+1) \simeq \mathbb{R} \times \mathbb{C}$.

(8) Does there exist a finite group G such that the Wedderburn decomposition of $\mathbb{C}G$ has the following form?
   (a) $M_2(\mathbb{C})$
   (b) $\mathbb{C} \times M_2(\mathbb{C})$
   (c) $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

23. Tutorial 12B (KNR 7)

Let $\lambda : \lambda_1 \geq \ldots \lambda_l \geq 1$ and $\mu : \mu_1 \geq \ldots \geq \mu_1 \geq 1$ be partitions of n. The set of semi-standard Young tableaux of shape λ and type μ is denoted by $\mathbb{K}_{\lambda\mu}$, and the cardinality of $\mathbb{K}_{\lambda\mu}$ by $K_{\lambda\mu}$. Observe the following:

(1) $K_{\lambda\lambda} = 1$.

(2) Let $\mu = 1 + 1 + \cdots + 1$ (n times). Then $\mathbb{K}_{\lambda\mu}$ is the set of standard Young tableaux of shape λ (each filled with numbers $1, \ldots, n$, each used exactly once) and $K_{\lambda\mu}$ is their number.

(3) $K_{\lambda\mu} \neq 0$ implies that λ *dominates* μ, that is, $l \leq m$ and $\lambda_1 + \cdots + \lambda_j \geq \mu_1 + \cdots + \mu_j$ for every $1 \leq j \leq l$. (The converse is also true, but it takes some proving.)

(4) Consider the total order—which for lack of a better term we call the *lexicographic order*—on partitions of n defined by: $\lambda \lneq \mu$ if $\lambda_j > \mu_j$ for the least j, $j \geq 1$, such that $\lambda_j \neq \mu_j$. Observe that $\lambda \leq \mu$ if λ dominates μ. So, by the previous item, $\lambda \leq \mu$ if $K_{\lambda\mu} \neq 0$.

(5) We define a square matrix K with non-negative integer entries as follows. Its rows and columns are both indexed by partitions arranged in increasing lexicographic order. The entry in the row corresponding to λ and the column corresponding to μ is $K_{\lambda\mu}$. Observe that:0

    K is an upper triangular non-negative integer matrix; its diagonal entries are all 1.

(6) Recall the statement of the RSK correspondence. Solution: There is a bijection:

$$\mathbb{M}_{\lambda\mu} \simeq \sqcup_{\nu \vdash n} \mathbb{K}_{\nu\lambda} \times \mathbb{K}_{\nu\mu}$$

Here $\mathbb{M}_{\lambda\mu}$ denotes the set of non-negative integer matrices of size $l \times m$ with row sums being $\lambda_1, \ldots, \lambda_l$ and column sums being $\mu_1, \ldots, \mu_m$. Since $\mathbb{K}_{\nu\lambda}$ is empty unless $\nu \leq \lambda$ and $\mathbb{K}_{\nu\mu}$ is empty unless $\nu \leq \mu$, the disjoint union on the right may well be taken only over those ν such that $\nu \leq \lambda$ and $\nu \leq \mu$.

(7) Let A be the diagonal $l \times l$ matrix with diagonal entries $\lambda_1, \ldots, \lambda_l$. Which ordered pair of SSYTs is attached to A by the RSK correspondence?

(8) Let T be the SSYT of shape λ with all entries in row j being j (for all j). Identify the $l \times l$ square matrix that is mapped by the RSK correspondence to the ordered pair (T, T).

(9) Observe that if A in $\mathbb{M}_{\lambda\mu}$ is mapped to (S, T) by the RSK correspondence, then A^t in $\mathbb{M}_{\mu\lambda}$ is mapped to (T, S).

(10) Let $M_{\lambda\mu}$ denote the cardinality of $\mathbb{M}_{\lambda\mu}$ (defined in the solution to the previous item). We define a square matrix M with non-negative integer entries as follows. Its rows and columns are both indexed by partitions arranged in increasing lexicographic order. The entry in the row corresponding to λ and the column corresponding to μ is $M_{\lambda\mu}$. Deduce from the RSK correspondence that:

$$M = K^t K \quad \text{where } K \text{ is the matrix defined in item (5) above}$$

(11) Let p be the number of partitions of n. Let C' be the $p \times 1$ matrix described as follows: the rows of C' are indexed by partitions arranged in increasing lexicographic order, and the entry in the row indexed by λ is the character of $\mathbb{C}X_\lambda$. By item (2) of Tutorial 11B, we have

$$\langle C'_\lambda, C'_\mu \rangle = \langle C'_\mu, C'_\lambda \rangle = \dim \mathrm{Hom}_G(\mathbb{C}X_\lambda, \mathbb{C}X_\mu) = M_{\lambda\mu} \quad \text{or more succinctly } \langle C', C'^t \rangle = M$$

Put $C = (K^t)^{-1} C'$. Observe that $(K^t)^{-1}$ is a lower triangular integer matrix (with possibly negative integers below the diagonal) with all diagonal entries equal to 1. Each entry of C is an integral linear combination of irreducible characters (since C' has this property). Moreover, we have the following (using the fact that $M = K^t K$ from item (10) above):

$$\begin{aligned}\langle C, C^t \rangle &= \langle (K^t)^{-1} C', C'^t K^{-1} \rangle = (K^t)^{-1} \langle C', C'^t \rangle K^{-1} \\ &= (K^t)^{-1} M K^{-1} = (K^t)^{-1} (K^t K) K^{-1} = \text{Identity}_{p \times p}\end{aligned}$$

Thus the entries of C are the irreducible characters possibly up to a sign factor. But since C'_λ is a character, and $\langle C'_\lambda, C_\lambda \rangle = \langle (K^t C)_\lambda, C_\lambda \rangle = \sum K_{\mu\lambda} \langle C_\mu, C_\lambda \rangle = K_{\lambda\lambda} = 1$, it follows that the entries of C are precisely the irreducible characters (not just up to sign).

Let V_λ denote the complex irreducible representation of $\mathfrak{S}_n$ with character C_λ.

(12) (Young's rule) Show that $\mathbb{C}X_\lambda = \sum_{\mu \vdash n} V_\mu^{\oplus K_{\mu\lambda}}$. (Hint: From $C' = K^t C$, we obtain $C'_\lambda = \sum_\mu K_{\mu\lambda} C_\mu$, which, since characters determine representatations, means $\mathbb{C}X_\lambda = \sum_\mu V_\mu^{\oplus K_{\mu\lambda}}$)

(13) Deduce from Young's rule that the dimension of the irreducible representation V_λ equals the number (denoted by f^λ) of the number of standard Young tableaux of shape λ (filled with numbers $1, \ldots, n$, each used exactly once). (Hint: Put $\lambda = 1 + 1 + \cdots + 1$ (n times). Then $\mathbb{C}X_\lambda$ is the regular representation of $\mathfrak{S}_n$. By Young's rule the multiplicity of V_μ in $\mathbb{C}X_\lambda$ equals $K_{\mu\lambda}$. But $K_{\mu\lambda}$ equals f^μ, and any irreducible representation occurs exactly as many times as its dimension in the regular representation.)

(14) Deduce that $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$. (Hint: We know that the order of a finite group equals the sum of the squares of the dimensions of its irreducible representations. Apply this to $\mathfrak{S}_n$ and use items (11) and (13) above.)

(15) The RSK correspondence restricted to permutation matrices gives a bijection onto ordered pairs of standard Young tableaux of the same shape. In particular, we obtain a combinatorial proof of the identity of item (14). (Hint: Put $\lambda = \mu = 1 + 1 + \cdots + 1$ (n times). Then $\mathbb{M}_{\lambda\mu}$ is the set of all $n \times n$ permutation matrices, and $\mathbb{K}_{\nu\lambda} = \mathbb{K}_{\nu\mu}$ is the set of all standard Young tableaux of shape ν.)