Separable algebras

[s:sepalg]

All algebras considered in this section are commmuative and associative over a field. Let K be a field and A such a K-algebra. We will use L to denote an (arbitrary) extension field of K; and $A_{(L)}$ to denote the L-algebra $A \otimes_K L$.

Étale algebras and separable algebras defined. We say that A is *diagonalizable* if A is isomorphic to a finite product of copies of the field K (in which case it is clearly finite dimensional over K). We say that A is

- *separable* if $A_{(L)}$ is reduced for every possible extension L of K;
- *étale* if $A_{(L)}$ is a diagonlizable L-algebra for some extension L.

These definitions apply in particular to the case when A is a field extension of K. Clearly A is étale if it is diagonalizable.

Some examples.

- The field K as an algebra over itself is separable since $K_{(L)} = L$.
- A polynomial algebra over K (in any number of indeterminates) is separable since $K[X_i]_{(L)} = L[X_i]$.
- A localization of a separable algebra is separable. In particular, a pure transcendental extension $K(X_i)$ (in any number of indeterminates) is separable. Proof: $(S^{-1}A)_{(L)} = S^{-1}(A_{(L)})$, and localization of a reduced algebra is reduced.
- As we will see later, any reduced algebra over a perfect field is separable. In particular, field extensions over perfect fields are separable.

Some basic observations.

- A product of separable algebras is separable (because $(\prod A_i)_{(L)} \hookrightarrow \prod(A_{i(L)})$, although this is not in general an isomorphism).
- A diagonalizable algebra is separable (proof: the algebra is a finite product copies of K).
- $A_{(E)} \subseteq A_{(F)}$ for extensions $E \subseteq F$; $B_{(L)} \subseteq A_{(L)}$ if $B \subseteq A$ as algebras.
- Subalgebras of separable algebras are separable (because $B_{(L)} \subseteq A_{(L)}$) .
- A is separable if and only if $A_{(E)}$ is a separable E-algebra for some extension E of K. Proof: $(A_{(E)})_{(L)} = A_{(L)}$, which proves the only if part; conversely, given an extension L of K, we can find a common extenstion F of both L and E, and then $A_{(L)} \subseteq A_{(F)} = (A_{(E)})_{(F)}$, so $A_{(L)}$ is reduced.
- An étale algebra is separable. Proof: Choose L such that $A_{(L)}$ is diagonalizable. In particular, $A_{(L)}$ is a separable L-algebra. Now apply the previous assertion.
- A finite dimensional separable algebra is étale. Proof: Let L be an algebraically closed extension of K. Then $A_{(L)}$, being a reduced finite dimensional L-algebra, is isomorphic to a finite product of copies of L.
- A direct limit of separable algebras is separable (because $(\varinjlim A_i)_{(L)} = \varinjlim A_{i(L)}$). In particular, an extension is separable if all finite subextensions are étale.

using fact: any vector space is flat

étale is separable; finite dimensional separable is étale (uses Structure theorem for Artin rings)

A simple but crucial observation and its consequences. The following observation is crucial although its proof is simple:

If A is a separable and B a reduced K-algebra, then $A \otimes_K B$ is reduced. Proof: Being reduced, B is a subalgebra of a product $\prod K_i$ of fields, which may be treated as extensions of K. Now $A \otimes_K \prod K_i \hookrightarrow \prod (A \otimes_K K_i)$, which is reduced by the separability of A.

separable tensor reduced is reduced

We deduce the following corollaries:

separable over separable is separable

separable tensor separable is separable

- If E is a separable extension of K and A a separable E-algebra, then A is a separable K-algebra. PROOF: We have $A_{(L)} = A \otimes_K L = (A \otimes_E E) \otimes_K L = A \otimes_E (E \otimes_K L) = A \otimes_K E_{(L)}$. Now A is a separable E-algebra and $E_{(L)}$ is reduced since E is a separable K-algebra. Thus $A \otimes_E E_{(L)}$ is reduced by the above observation.
- If A are B are separable K-algebras, then so is $A \otimes_K B$. PROOF: $(A \otimes_K B) \otimes_K L = A \otimes_K (B \otimes_K L)$, and the latter is reduced since A is separable and $B_{(L)}$ reduced.

Notable however is the following:

- EXAMPLE: Let $K \subseteq L \subseteq M$ be fields with M separable over K. Then of course L is separable over K (being a subextension of a separable extension), but M need not be separable over L. For example, let K be a field of positive characteristic p, $L = K(X^p)$, and $M = K(X)$, where X is an indeterminate. Later on it's shown that if further L is algebraic over K, then M is separable over L.

An elemental characterization of separability in positive characteristic. We assume in this subsection that the characteristic p of K is positive. Observe that $K^{p^f} := \{\lambda^{p^f} \mid \lambda \in K\}$ is a subfield of K and $A^{p^f} := \{a^{p^f} \mid a \in A\}$ a K^{p^f}-subalgebra of A. Suppose that A is reduced. Then the map $x \mapsto x^{p^f}$ (f-th power of the Frobenius morphism) is an isomorphism of rings $A \to A^{p^f}$; a subset $\{a_i\}$ of A is K-linearly independent (respectively, a K-basis of A) if and only if $\{a_i^{p^f}\}$ is K^{p^f}-linearly independent (respectively, a K^{p^f}-basis of A^{p^f}); further, for L an extension field of K, the map f-th power of the Frobenius gives an isomorphism of rings $A \otimes_K L$ and $A^{p^f} \otimes_{K^{p^f}} L^{p^f}$.

The following are equivalent for a K-algebra A:

1. A is separable.
2. $A^{p^f} \otimes_{K^{p^f}} K$ is reduced for every $f \geq 0$.
3. A is reduced and $A^p \otimes_{K^p} K$ is reduced.
4. A is reduced and the natural map $A^p \otimes_{K^p} K \to A$ given by $a^p \otimes \lambda \mapsto \lambda^p$ is an injection.
5. For every K-linearly independent subset $\{a_i\}$ of A, the subset $\{a_i^p\}$ continues to be K-linearly independent.
6. There exists a K-basis $\{a_i\}$ of A such that $\{a_i^p\}$ is K-linearly independent.

PROOF: (1)$\Rightarrow$(2): A is reduced and so is $A \otimes_K K^{p^{-f}}$. But, by our observation above, $A \otimes_K K^{p^{-f}}$ is isomorphic to the ring $A^{p^f} \otimes_{K^{p^f}} K$.

(2)$\Rightarrow$(3): set $f = 0$ and $f = 1$.

(3)$\Rightarrow$(4): If $x = \sum a_i^p \otimes \lambda_i \mapsto 0$, then $\sum \lambda_i a_i^p = 0$, which implies $x^p = \sum (a_i^p)^p \otimes \lambda_i^p = ((\sum a_i^p \lambda_i)^p) \otimes 1 = 0$. Since $A^p \otimes_{K^p} K$ is reduced, this means $x = 0$.

(4)$\Rightarrow$(5): As already remarked, K-linear independence of $\{a_i\}$ is equivalent to the K^p-linear independence of $\{a_i^p\}$. If $\sum \lambda_i a_i^p = 0$ ($\lambda_i \in K$ almost all zero), then $\sum a_i^p \otimes \lambda_i$ in $A^p \otimes_{K^p} K$ maps to zero in A, hence is zero by hypothesis. By the K^p-linear indenpendence of $\{a_i^p\}$, we conclude that the λ_i all vanish.

(5)⇒(6): Obvious.

(6)⇒(1): Let L be an arbitrary extension of K. To show that $A_{(L)}$ is reduced, it is enough to show that any element x of $A_{(L)}$ such that $x^p = 0$ is itself 0. Let x be such an element and write x (uniquely) as $\sum a_i \otimes l_i$. We have $x^p = \sum a_i^p \otimes l_i^p$. Since the a_i^p are K-linearly indenpendent, we conclude that $l_i^p = 0$, and so $l_i = 0$, and $x = 0$. □

Condition (2) is equivalent to $A \otimes_K K^{p^{-\infty}}$ being reduced; and condition (3) to $A \otimes_K K^{p^{-1}}$ being reduced. Thus for A to be separable it is enough that $A \otimes_K K^{p^{-\infty}}$ or even $A \otimes_K K^{p^{-1}}$ is reduced.

Separability (or lack thereof) of extension fields. Recall that a polynomial in one indeterminate over K is *separable* if it has no repeated roots, or, equivalently (and more intrinsically), if it is coprime to its derivative.

- A simple extension $K[\alpha]$ is separable over K if and only if the minimal polynomial of α over K is separable.
- An algebraic extension is separable if and only if the minimal polynomial over K of every element of the extension field is separable.

  Proof: For the only if part, observe that for any element α of the extension field, the simple extension $K[\alpha]$ is separable (being a subalgebra of a separable algebra), so the minimal polynomial of α is separable by the previous item.

  For the converse, it is enough to show that every finite extension is separable since any extension is the direct limit of its finite extensions. Given a finite extension $K[\alpha_1, \ldots, \alpha_n]$, consider the chain $K \subseteq K[\alpha_1] \subseteq K[\alpha_1][\alpha_2] \subseteq K[\alpha_1, \alpha_2][\alpha_3] \subseteq \ldots \subseteq K[\alpha_1, \ldots, \alpha_{n-1}][x_n]$. The minimal polynomial of α_j over $K[\alpha_1, \ldots, \alpha_{j-1}]$ being a factor of that over K, it is separable. But separable over separable is separable (as already shown), so $K[\alpha_1, \ldots, \alpha_n]$ is separable over K.

- Any extension in characteristic 0 is separable.

  Proof: Any irreducible polynomial over a field of characteristic 0 being separable, the separability in case the extension is algebraic follows from the previous item. For a general extension E, let $\{x_i\}$ be a transcendence basis. Then $K(x_i)$ is separable over K (for, as already observed, pure transcendental extensions are separable); on the other hand, E being algebraic over $K(x_i)$, it is separable over $K(x_i)$ by what we just observed. Finally, separable over separable is separable (as already observed above).

- (Maclane's criterion) A field extension $K \subseteq L$ in positive characteristic p is separable if and only if L^p and K are K^p-linearly disjoint.

  Proof: Let $\{a_i\}$ be a subset of L such that $\{a_i^p\}$ is K^p-linearly independent. As already observed, this is equivalent to $\{a_i\}$ being K-linearly independent. Now the result follows from the equivalence of conditions (1) and (5) in the elemental characterization above applied to L.

Maclane's criteria

- (Maclane's Criterion; second version) Let $K \subseteq L \subseteq \Omega$ be fields of positive characteristic p with Ω perfect. Then L is separable over K if and only if it is K-linearly disjoint from the perfect closure $K^{p^{-\infty}}$ of K inside Ω.

  Proof: Suppose that L is separable and let $\{l_i\}$ be a K-linearly independent subset of L. Then, by repeatedly applying property (5) above, we conclude that $\{l_i^{p^f}\}$ is K-linearly independent (for any $f \geq 0$). This means that $\{l_i\}$ are $K^{p^{-f}}$-linearly independent (for all f): we can see this by just raising any $K^{p^{-f}}$-linear relation among the l_i to the power p^f.

  Conversely, linear disjointness over K of L and $K^{p^{-\infty}}$ implies that of L and $K^{p^{-1}}$. Thus any K-basis $\{l_i\}$ of L is $K^{p^{-1}}$-linearly independent. As is easily seen, this linear independence is equivalent to the K-linear independence of $\{l_i^p\}$. And so the equivalent condition (6) above for separability holds.

- If L is a separable field extension of K, then $L \otimes_K K^{p^{-\infty}}$ is a field; further, if L is algebraic over K, then $L \otimes_K K^{p^{-\infty}}$ is a perfect closure of L.

  Proof: Set $\Omega = L^{p^{-\infty}}$ and apply the last criterion. By the K-linear disjointness of L and $K^{p^{-\infty}}$, the natural map $L \otimes K^{p^{-\infty}} \to L^{p^{-\infty}}$ is an injection. So $L \otimes_K K^{p^{-\infty}}$ is a domain. Being caught between L and $L^{p^{-\infty}}$ it is a field and algebraic over L. If, moreover, L is algebraic over K, then $L \otimes_K K^{p^{-\infty}}$ is algebraic over $K^{p^{-\infty}}$ and is therefore perfect. Since it contains L, it must be all of $L^{p^{-\infty}}$.

Separability over perfect base fields.

reduced over perfect is separable; any extension over a perfect field is separable

- Any reduced algebra over a perfect field is separable. In particular, any extension over a perfect field is separable.

  Proof: Since reduced tensor separable is reduced (proved above), it is enough to show that any extension L over K is separable when K is perfect. It has been shown above that any extension in characteristic 0 is separable, so we may assume that the characteristic is positive. Consider condition (3) of the elemental characterization of separability above. Since $K^p = K$ (because K is perfect), we have $A^p \otimes_{K^p} K \cong A^p$ which being a subalgebra of A is reduced, and so condition (3) holds.

- For A to be separable, it is enough that $A_{(L)}$ be reduced for some perfect extension L.

  Proof: If $A_{(L)}$ is reduced, then $A_{(L)}$ is L-separable, which easily implies that A is K-separable (as already noted above).

Complements. Let $K \subseteq L \subseteq M$ be extension fields with M separable over K and L algebraic over K. Then M is separable over L. The hypothesis that L is algebraic over K cannot be omitted as the example already considered shows.

Proof: If the characteristic is zero, then any field extension is separable. So suppose now that it is positive. It is enough to show that $M \otimes_L L^{p^{-\infty}}$ is reduced (as just seen above). We have $L^{p^{-\infty}} = L \otimes_K K^{p^{-\infty}}$ (as also seen above), so that $M \otimes_L L^{p^{-\infty}} = M \otimes_L L \otimes_K K^{p^{-\infty}} = M \otimes_K K^{p^{-\infty}}$. The latter is reduced since M is separable over K.