Let K be a commutative ring with identity, A a commutative K-algebra, and M an A-module. We will be especially interested in the case when K is a field and A a field extension of K. A *K-derivation* of A to M is a K-linear map $d : A \to M$ satisfying the *Liebniz rule*: $d(ab) = adb + bda$. The set $\mathrm{D}er_K(A, M)$ of K-derivations of A to M is an A-submodule of $\mathrm{Hom}_K(A, M)$. If we speak of just a *derivation* we mean a $\mathbb{Z}$-derivation. Derivations defined

The module $\Omega_K(A)$ of differentials. There exists an A-module denoted $\Omega_K(A)$, the *module of differentials*, and a *universal* K-derivation $d = d_{A/K}$ from A to $\Omega_K(A)$ with the following properties:

- Universal property: $\mathrm{Hom}(\Omega_K(A), M) \to \mathrm{D}er_K(A, M)$ defined by $\phi \mapsto \phi \circ d$ is a bijection.
- The image of d generates $\Omega_K(A)$ as an A-module.

Goal of this section. We consider $\Omega_K(L)$ where $K \subseteq L$ are extension fields and aim to relate properties of the extension with those of the L-vector space $\Omega_K(L)$.

Some elementary properties and examples.

(1) The image of K in A under any K-derivation is zero. In particular, the image of K in $\Omega_K(A)$ under the universal derivation is zero. Proof: $d1 = d(1.1) = 1d1 + 1d1$ and so $d1 = 0$; now use K-linearity of d.

(2) Let D be a K'-derivation of A to M, where $K' \to K$ is a map of commutative rings. Then D is a K-derivation if and only if the image of K under D vanishes. Proof: The only if part is the previous item. For the if part, we need only show K-linearity of D. For λ in K and a in A, we have $D(\lambda a) = \lambda Da + aD\lambda$ by the Liebiz rule, but $D\lambda = 0$ by hypothesis, so D is K-linear.

(3) Any derivation vanishes on the image of $\mathbb{Z}$ in K; if K is a field, then any derivation is a P-derivation where P is the prime subfield. Proof: The first assertion is a special case of the first item. The second follows from the first and second items.

(4) A K-derivation on A is determined by its vaules on any set of K-algebra generators of A. In particular, a derivation on A is determined by its values on K and on a set of K-algebra generators of A.

(5) Let A be a polynomial ring $K[X_i \,|\, i \in I]$ over K (in any number of variables) and V an A-module. Given a derivation $\Delta : K \to V$ and an arbitrary $\{v_i\}_{i \in I}$ of elements of V, there exists a unique derivation D that extends Δ and maps X_i to v_i: more precisely, the value of D on any polynomial f is given by: $f^\Delta(\underline{x}) + \sum_{i \in I} \frac{\partial f}{\partial X_i}(\underline{x}) v_i$, where, for $f(\underline{x}) = \sum_\alpha a_\alpha X^\alpha$, we have $f^\Delta(\underline{x}) := \sum_\alpha X^\alpha \Delta(a_\alpha)$. In particular, there exists a unique K-derivation that maps X_i to v_i. This last assertion is equivalent to saying that $\Omega_K(A)$ is a free A-module on the generators dX_i, $i \in I$.

(6) Let A be a quotient of the polynomial ring $K[X_i \,|\, i \in I]$ by an ideal $\mathfrak{a}$. Let $\{f_\lambda\}$ be a family of polynomials generating the ideal $\mathfrak{a}$ and let V be an A-module. We have a "pull-back" map $\mathrm{D}er(A, V) \to$

$\mathrm{D}er(K[X_i], V)$. The image of this map consists precisely of those elements of $\mathrm{D}er(K[X_i], V)$ that vanish on $\mathfrak{a}$.

Let $\Delta : K \to V$ be a derivation, and $\{v_i\}_{i\in I}$ a family of elements of V. Then there exists a (necessarily unique) derivation $D : A \to V$ that extends Δ and mapping X_i to v_i if and only if, for every λ, $f_\lambda^\Delta(\underline{x}) + \sum_{i\in I} \frac{\partial f_\lambda}{\partial X_i}(\underline{x})v_i = 0$. PROOF: As seen above, $f^\Delta(\underline{x}) + \sum_{i\in I} \frac{\partial f}{\partial X_i}(\underline{x})v_i$ is the value on a polynomial f of any derivation $D : K[X_i] \to V$ that extends Δ and is subject to $X_i \mapsto v_i$. To prove the "only if" part, let D be such a derivation. Denote by D also the pull back of D to $K[X_i]$. Since D vanishes on f_λ (because the image of f_λ in A vanishes), the equations are necessary. For the "if" part, let D be the unique extension to $K[X_i]$ of Δ such that $DX_i = v_i$. We will show that if the equations hold, then D vanishes on $\mathfrak{a}$. Then D will factor through A, and we'll be done. A general element x of $\mathfrak{a}$ is of the form $\sum g_\lambda f_\lambda$. Since D is a derivation, we have $Dx = \sum(f_\lambda Dg_\lambda + g_\lambda Df_\lambda)$. The equations mean that the Df_λ and hence also the second terms in this sum vanish. The first terms in the sum vanish since the Dg_λ belong to V which is an A-module and the images of f_λ in A vanish.

(7) With notation as in the item (5), assume that K is a field, let L denote the quotient field of A, let W be an L-vector space, and $D : A \to W$ a derivation. Then there exists a unique extension of D to a derivation $\tilde{D}$ from L to W. In fact, for f, g in A with $g \neq 0$, we have $\tilde{D}\frac{f}{g} = \frac{gDf - fDg}{g^2}$, by the "quotient rule". Thus $\Omega_K(L) = L \otimes_A \Omega_K(A)$, and $\{dX_i\}$ forms an L-basis for $\Omega_K(L)$.

The canonical exact sequence. Let B be another K-algebra and $A \to B$ a K-algebra homomorphism. By the universal properties of $\Omega_K(A)$ and $\Omega_K(B)$ respectively, there exists a natural B-module morphism $\beta : \Omega_K(B) \to \Omega_A(B)$ and a natural A-module morphism $\alpha : \Omega_K(A) \to \Omega_K(B)$. Since $\Omega_K(B)$ is a B-module, the map α extends uniquely to a map of B-modules $\Omega_K(A) \otimes_A B \to \Omega_K(B)$ also denoted α. Consider the following sequence of B-module maps:

$$\Omega_K(A) \otimes_A B \xrightarrow{\alpha} \Omega_K(B) \xrightarrow{\beta} \Omega_A(B) \to 0 \tag{6}$$

We claim that this is exact. To prove the claim it is enough to show that the resulting sequence after applying $\mathrm{Hom}_B(\cdot, V)$ is exact for any B-module V. The application of $\mathrm{Hom}_B(\cdot, V)$ results in:

$$\mathrm{D}er_K(A, V) \xleftarrow{\alpha^*} \mathrm{D}er_K(B, V) \xleftarrow{\beta^*} \mathrm{D}er_A(B, V) \leftarrow 0 \tag{7}$$

where α^* is the natural restriction map and β^* is the natural inclusion. From the first two items in the list above, we conclude that the kernel of α^* is the image of β^*. Thus (7) and so also (6) is exact.

Suppose that B is a field. Then a sequence of B-modules is exact if and only if the dual sequence is exact. Thus the exactness of (6) is equivalent to that of

$$\mathrm{D}er_K(A, B) \xleftarrow{\alpha^*} \mathrm{D}er_K(B, B) \xleftarrow{\beta^*} \mathrm{D}er_A(B, B) \to 0 \tag{8}$$

Moreover the injectivity of α in (6) is equivalent to the surjectivity of α^* in (8). If these conditions hold, then the α^* in (7) is surjective for every V.