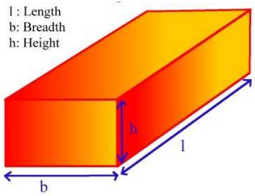
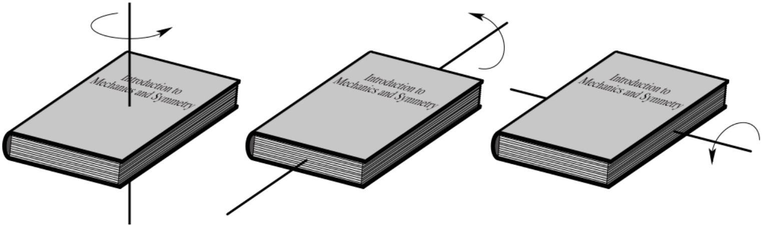
How to stably spin a cuboid

Sushmita Venugopalan

July 5, 2016

Spinning a cuboid, or any rigid body





- Watch video.

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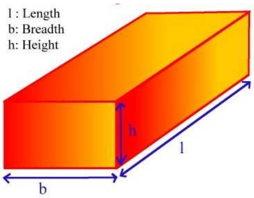

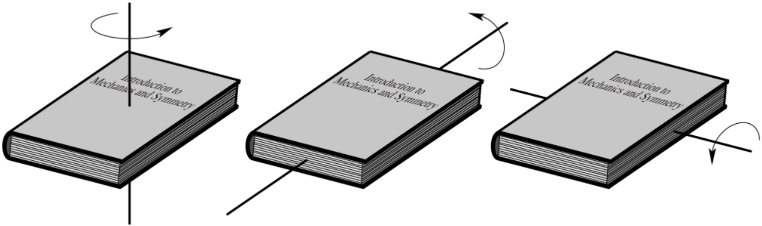

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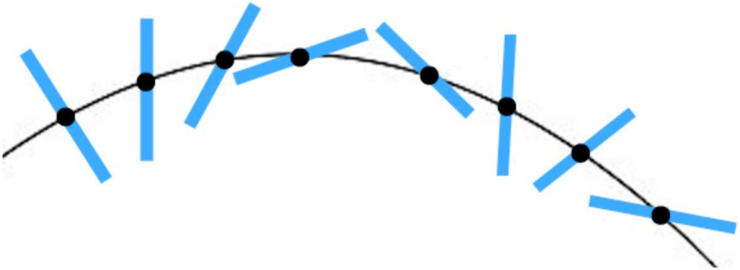
Splitting up rotation and translation



- The motion of a rigid body can be described as the super-imposition of translation and rotation about the center of mass.
- In the rest of the talk we assume that the center of mass is fixed, and the body rotates about the center of mass. Further, there is no force acting on the body after it is given an initial spin.

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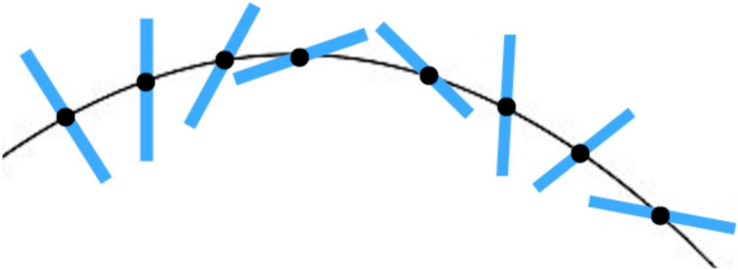

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Angular Momentum

- Angular momentum of a point mass about an axis of rotation is a vector that points in the direction of the axis of rotation.
- Its magnitude is *momentum times distance from axis*, OR
- Angular momentum $L = mr_{axis}^2\overline{\omega}$, where m =mass, r_{axis} = distance from the axis and ω=magnitude of angular velocity.
- For a rigid body Q, the angular momentum is $(\int r_{axis}^2 \mathrm{d}m)\omega$.
- The quantity $\int_Q r_{axis}^2 \mathrm{d}m$ is the *moment of inertia* of the rigid body about the axis under consideration.

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It is a conserved quantity!

- The angular momentum about an axis is conserved if there is zero torque about that axis.
- Watch video upto time 0:45.

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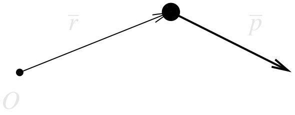
Breaking angular momentum into components

- In our problem the axis of rotation is not fixed - for example in the case when the box tumbles out of its orbit.
- Then, one considers angular momentum of a body about a **point** (as opposed to axis).
- Angular momentum of a point mass about a point O is the vector cross product

$$\begin{aligned} \overline{L} &= \overline{r} \times \overline{p} \\ &= m\overline{r} \times (\overline{r} \times \overline{\omega}). \end{aligned}$$



Position of particle=$\overline{r}$

Linear momentum=$\overline{p}$

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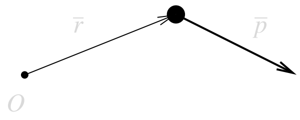

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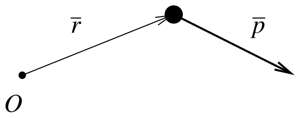

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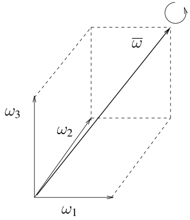
- For a rigid body, angular momentum can be computed component-wise, along the principal axes of the body.
- First split up angular velocity into components.



$\overline{\omega} = \omega_1 \hat{x} + \omega_2 \hat{y} + \omega_3 \hat{z}$

- Suppose I_1, I_2 and I_3 are moments of inertia along the 3 axes. Then, angular momentum is

$$L = L_1 \hat{x} + L_2 \hat{y} + L_3 \hat{z}, \quad L_i = I_i \omega_i, \quad i = 1, 2, 3.$$

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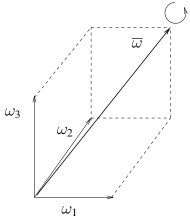

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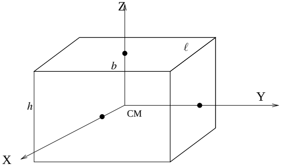
The case of the cuboid

- For the cuboid, the three principal axes at the center of mass are the lines parallel to the length, width and the height.



- The moments of inertia are

$$I_1 = \frac{1}{12}(b^2 + h^2), \quad I_2 = \frac{1}{12}(h^2 + l^2), \quad I_1 = \frac{1}{12}(l^2 + b^2).$$

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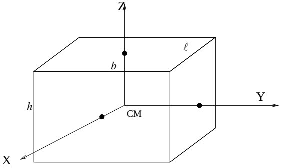

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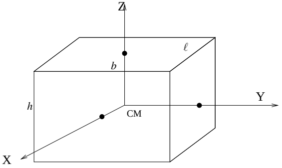

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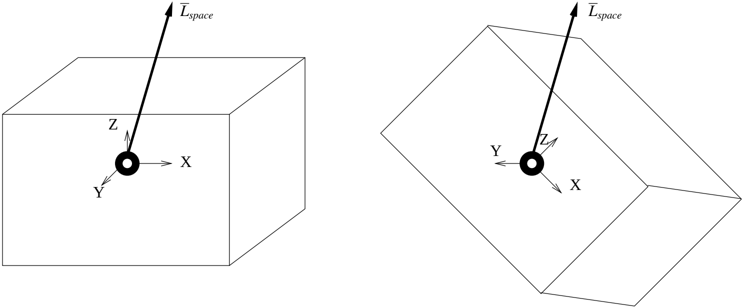
The right frame of reference

- If you observe from the ground, the standard axes will not be aligned with l, b and h and the above formula is no good!
- For observing the motion, we choose a coordinate axis that is fixed in the cuboid, and hence rotates with the cuboid.



Euler's equation

- The rate of change of angular momentum in the space frame and in the body frame is related by *Euler's equation*.

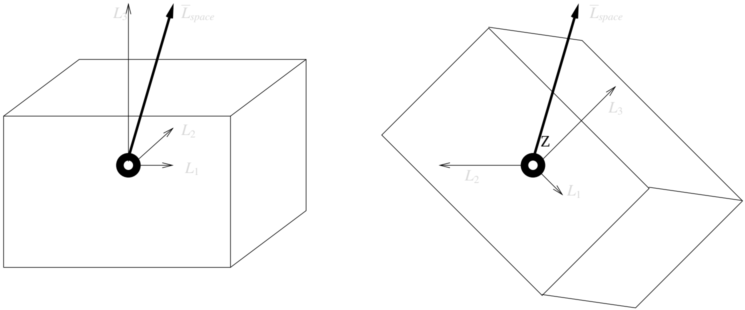
$$\frac{d}{dt}\overline{L}_{body} = \frac{d}{dt}\overline{L}_{space} + \overline{L}_{body} \times \overline{\omega}.$$

- By conservation of angular momentum $\frac{d}{dt}\overline{L}_{space} = 0$. So, the equation reduces to $\frac{d}{dt}\overline{L}_{body} = \overline{L}_{body} \times \overline{\omega}$.
- Setting $\overline{L}_{body} = \overline{L} = (L_1, L_2, L_3)$, the Euler's equation take the form

$$\begin{aligned}
\frac{d}{dt}L_1 &= (I_2 - I_3)\omega_2\omega_3 \\
\frac{d}{dt}L_2 &= (I_3 - I_1)\omega_3\omega_1 \\
\frac{d}{dt}L_3 &= (I_1 - I_2)\omega_1\omega_2
\end{aligned}$$

Wrapping head around the change of co-ordinates

- The angular momentum vector $\overline{L}_{space}$ is constant.
- But as the co-ordinate system rotates, its components along the body axes change over time.



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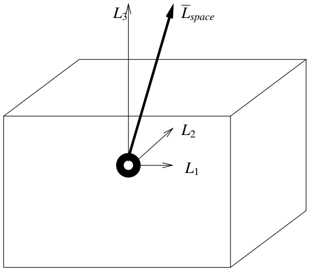
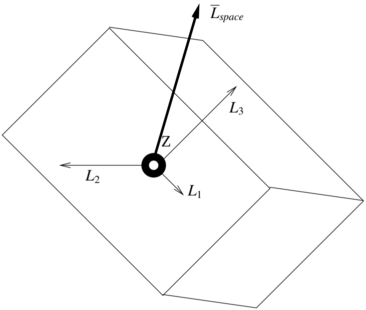




Evolution equation

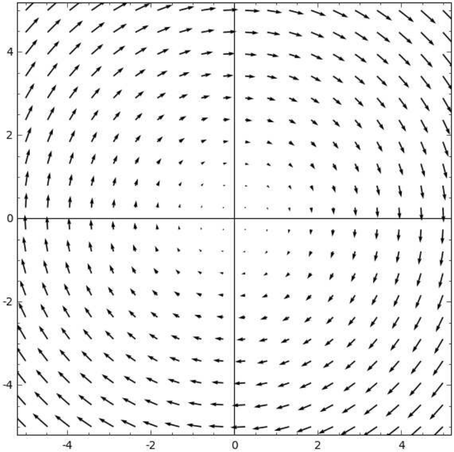
- Using $L_1 = I_1\omega_1, L_1 = I_1\omega_1, L_1 = I_1\omega_1$, we further re-write

$$\frac{d}{dt}L_1 = (\frac{1}{I_3} - \frac{1}{I_2})L_2L_3$$
$$\frac{d}{dt}L_2 = (\frac{1}{I_1} - \frac{1}{I_3})L_3L_1$$
$$\frac{d}{dt}L_3 = (\frac{1}{I_2} - \frac{1}{I_1})L_1L_2$$

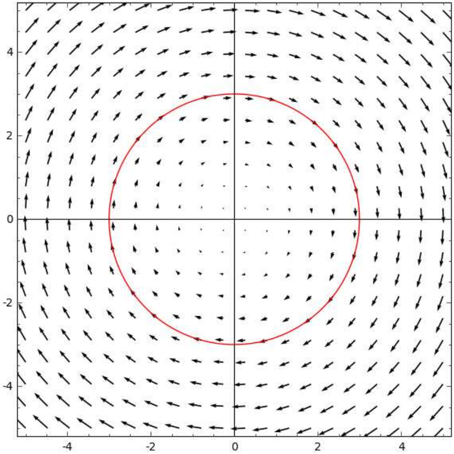
- The above system of differential equations is an example of an *evolution equation*. If one knows what is (L_1, L_2, L_3) at time $t = 0$, one can uniquely determine (L_1, L_2, L_3) as a function of time t.

Examples of evolution equation

Consider the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x$. The system can be represented as a *vector field* $\overline{F}$ in the x-y plane. The vector at the point (x, y) is $\overline{F}_{(x,y)} = (y, -x)$.



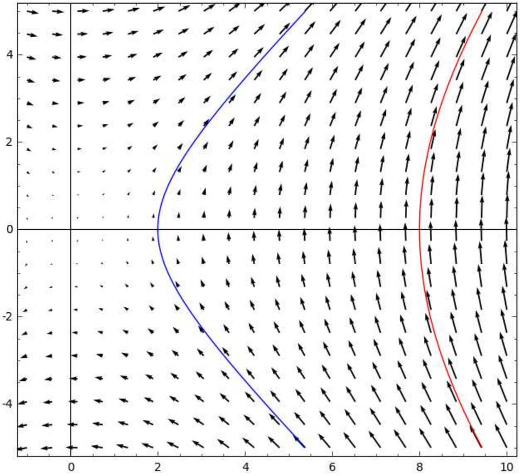
Given an *initial condition*, say $(x(0), y(0)) = (2, 0)$, there is a unique solution for the system of differential equations, namely $(x(t), y(t)) = (2\cos(t), 2\sin(t))$. A solution is a parametrized curve $(x(t), y(t))$ whose velocity vector at time t is $\overline{F}(x(t), y(t))$.



Examples of evolution equation

Solution of the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$, with

- initial condition $(x(0), y(0)) = (2, 0)$ is the blue curve, and with
- initial condition $(x(0), y(0)) = (8, 0)$ is the red curve.



Equilibrium points

- (Some calculations were done on the board. We studied the various trajectories of the saddle equilibrium in the second example. The discussion led us to the following conclusions.)
- The interesting activity in an evolution equation happens near *equilibrium points*, i.e. points where the vector field is zero.
- In the first example, the equilibrium point is of type *circle*. It is a stable equilibrium.
- In the second example, the equilibrium point is of type *saddle*. It is an unstable equilibrium.

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Understanding the phase space of $\overline{L}_{body}$

- In our problem, at the instant of throwing $(L_1(0), L_2(0), L_3(0))$ is determined. The solution to the evolution equation $(L_1(t), L_2(t), L_3(t))$ is a curve in 3-dimensional space.
- What is the curve in the phase space corresponding to stable rotation about the length axis?
- What is the curve in the phase space corresponding to stable rotation about the height axis?
- What if the rotation axis rotates about the length axis?
- Any guesses about the unstable axis?

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Answers

- What is the curve in the phase space corresponding to stable rotation about the length axis? It is the constant curve $(L_1(t), L_2(t), L_3(t)) = (L_1(0), 0, 0)$.
- What is the curve in the phase space corresponding to stable rotation about the height axis? It is the constant curve $(L_1(t), L_2(t), L_3(t)) = (0, 0, L_3(0))$.
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- Any guesses about the unstable axis? Will be answered later.

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Can you think of any expressions involving L_1, L_2, L_3 that stay constant?

A constraint on the solution curves

- The size of the vector (L_1, L_2, L_3) is constant because it is just obtained by rotating $\overline{L}_{space}$. So

$$L_1^2 + L_2^2 + L_3^2 = constt.$$

- Any solution curve lies on a sphere $L_1^2 + L_2^2 + L_3^2 = constt.$

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A constraint on the solution curves:alternate method

In the evolution equation

$$\frac{d}{dt}L_1 = (\frac{1}{I_3} - \frac{1}{I_2})L_2L_3 \tag{1}$$

$$\frac{d}{dt}L_2 = (\frac{1}{I_1} - \frac{1}{I_3})L_3L_1 \tag{2}$$

$$\frac{d}{dt}L_3 = (\frac{1}{I_2} - \frac{1}{I_1})L_1L_2 \tag{3}$$

Multiply L_1, L_2, L_3 to the equations (1), (2) and (3) respectively, and add to obtain

$$L_1\frac{d}{dt}L_1 + L_2\frac{d}{dt}L_2 + L_3\frac{d}{dt}L_3 = 0 \implies L_1^2 + L_2^2 + L_3^2 = constt.$$

Another constraint on the solution curves

In the evolution equation

$$\frac{d}{dt}L_1 = (\frac{1}{I_3} - \frac{1}{I_2})L_2L_3 \tag{4}$$

$$\frac{d}{dt}L_2 = (\frac{1}{I_1} - \frac{1}{I_3})L_3L_1 \tag{5}$$

$$\frac{d}{dt}L_3 = (\frac{1}{I_2} - \frac{1}{I_1})L_1L_2 \tag{6}$$

Multiply $\frac{L_1}{2I_1}$, $\frac{L_2}{2I_2}$, $\frac{L_3}{2I_3}$ to the equations (4), (5) and (6) respectively, and add to obtain

$$\frac{L_1}{2I_1}\frac{d}{dt}L_1+\frac{L_2}{2I_2}\frac{d}{dt}L_2+\frac{L_3}{2I_3}\frac{d}{dt}L_3 = 0 \implies \frac{1}{2I_1}L_1^2+\frac{1}{2I_2}L_2^2+\frac{1}{2I_3}L_3^2 = constt.$$

The expression in the last equation is exactly the kinetic energy of the spinning cuboid!

Equilibrium points

- What are the equilibrium points of the evolution equation

$$\frac{d}{dt}L_1 = (\frac{1}{I_3} - \frac{1}{I_2})L_2L_3,$$
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- On any sphere, the equilibrium points are intersecttion with the X, Y and Z axes. (Same for ellipsoid).

- The solution curve $(L_1(t), L_2(t), L_3(t))$ lies
  - on the sphere $L_1^2 + L_2^2 + L_3^2 = c_1$ and
  - on the ellipsoid $\frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}L_2^2 + \frac{1}{2I_3}L_3^2 = c_2$.
- Here the constants are $c_1 := L_1^2(0) + L_2^2(0) + L_3^2(0)$ and $c_2 := \frac{1}{2I_1}L_1^2(0) + \frac{1}{2I_2}L_2^2(0) + \frac{1}{2I_3}L_3^2(0)$.
- What is the intersection of an ellipsoid and sphere going to look like? Remember that the axes of the ellipse are pairwise unequal.

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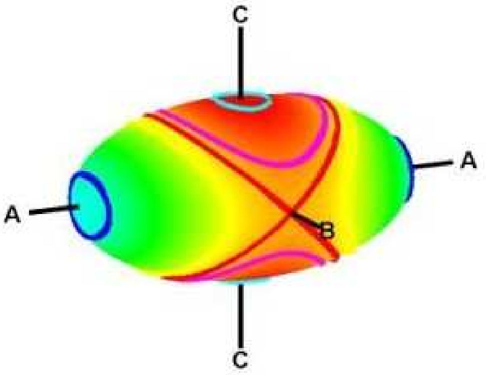
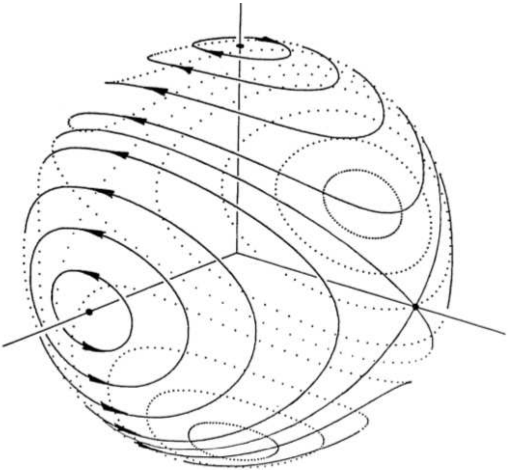
- What is the intersection of an ellipsoid and sphere going to look like? Remember that the axes of the ellipse are pairwise unequal.

- Consider a fixed ellipse $\frac{1}{2I_1}L_1^2 + \frac{1}{2I_2}L_2^2 + \frac{1}{2I_3}L_3^2 = c_2$.
- For different values of c_1, the intersection with the sphere $L_1^2 + L_2^2 + L_3^2 = c_1$ is given a different colour. Blue for the maximum value of c_1 and red for the minimum value of c_1.



The evolution trajectories



Credit:The Tumbling Box, Susan Jane Colley, The American Mathematical Monthly, Vol. 94, No. 1 (Jan., 1987), pp. 62-68.