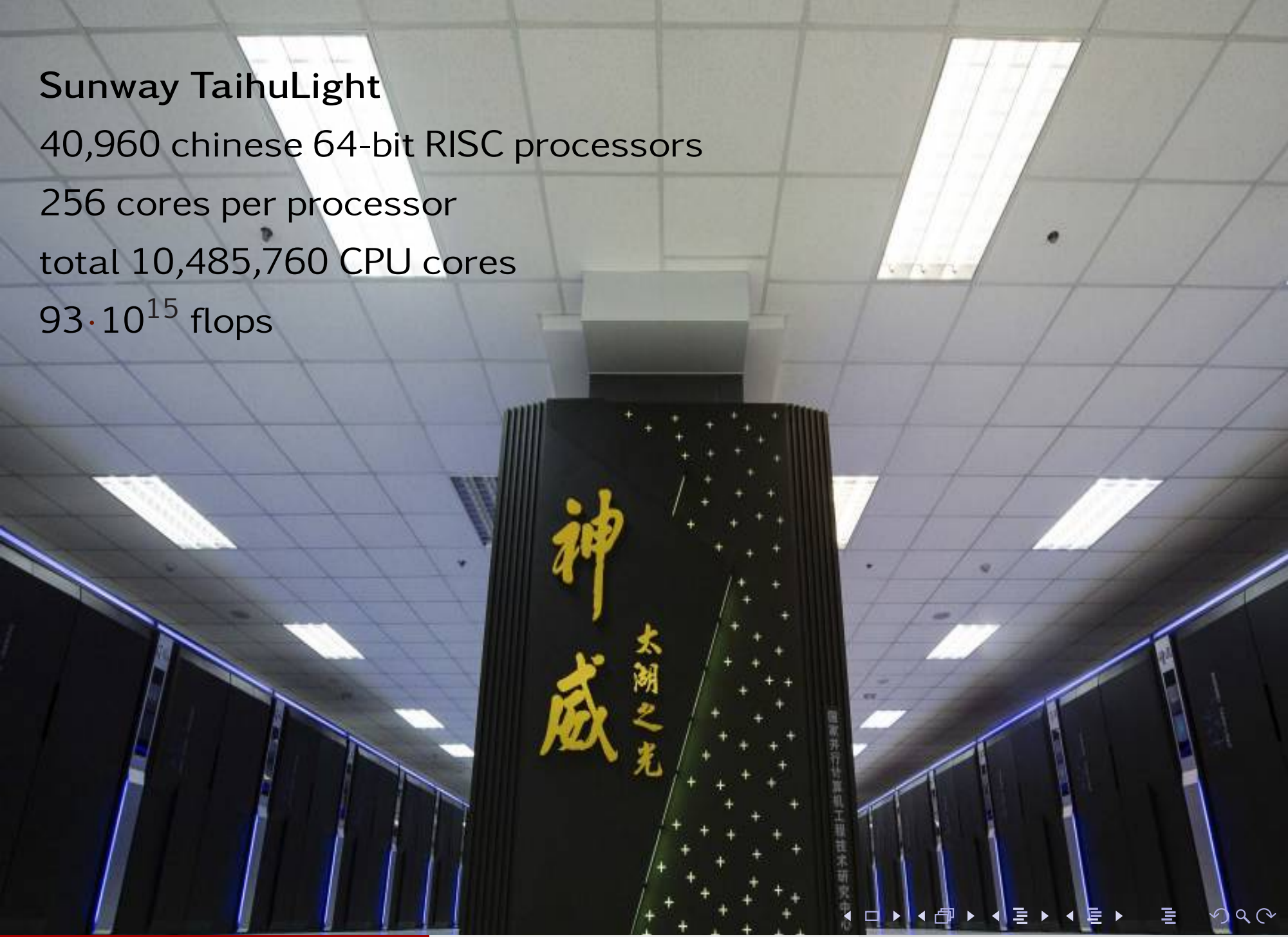
On Problems that Computers will Never Solve



Sunway TaihuLight

40,960 chinese 64-bit RISC processors

256 cores per processor

total 10,485,760 CPU cores

$93 \cdot 10^{15}$ flops

Let's try a puzzle

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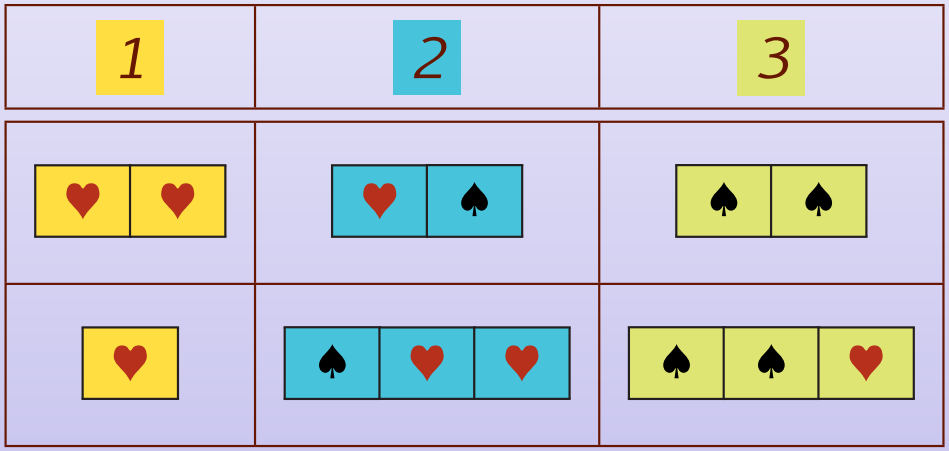

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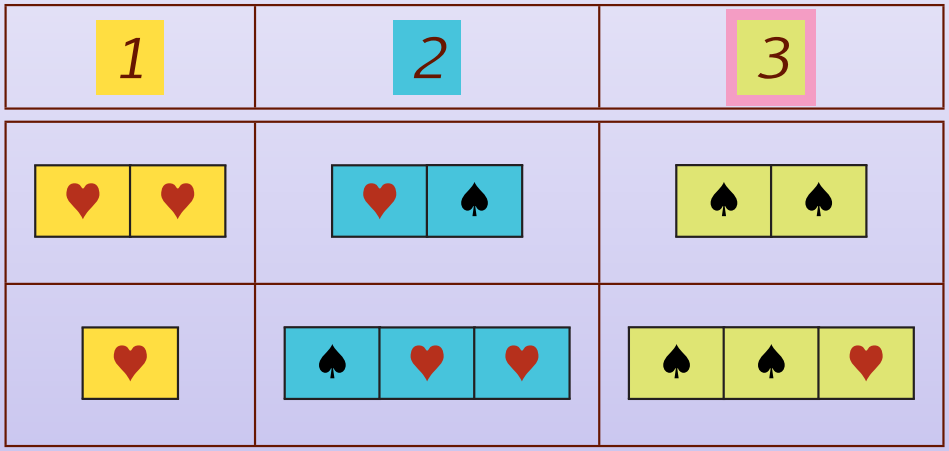

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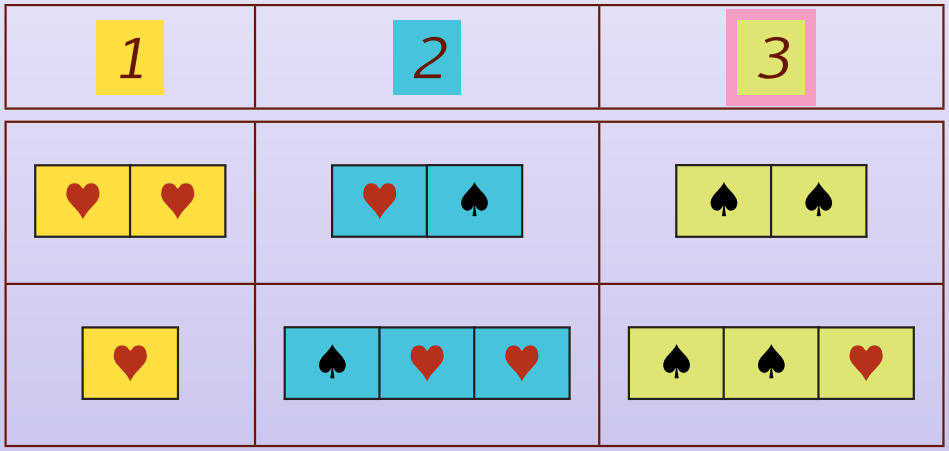

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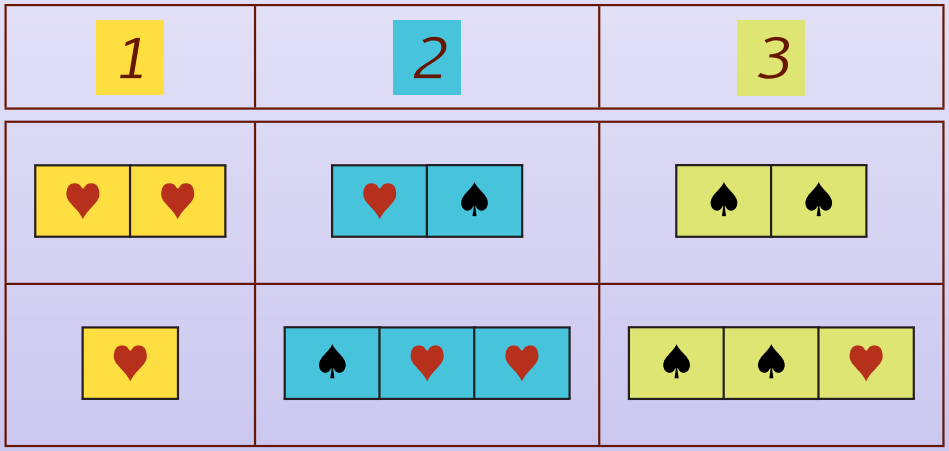

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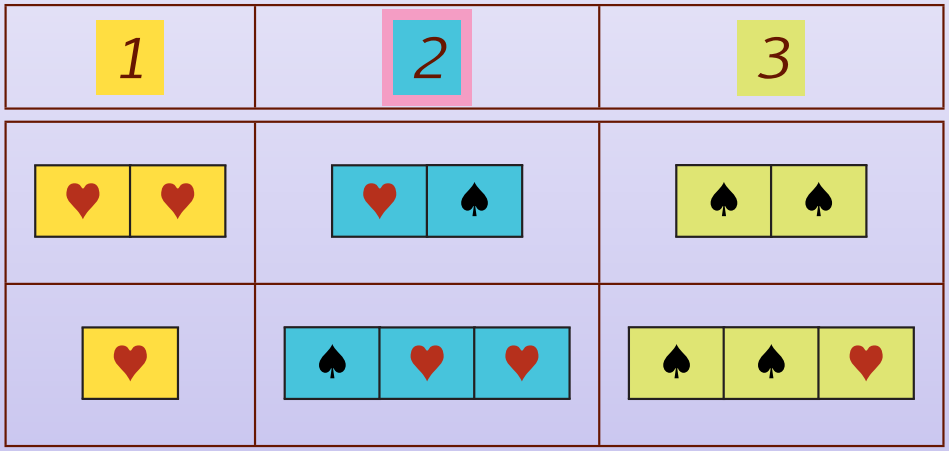

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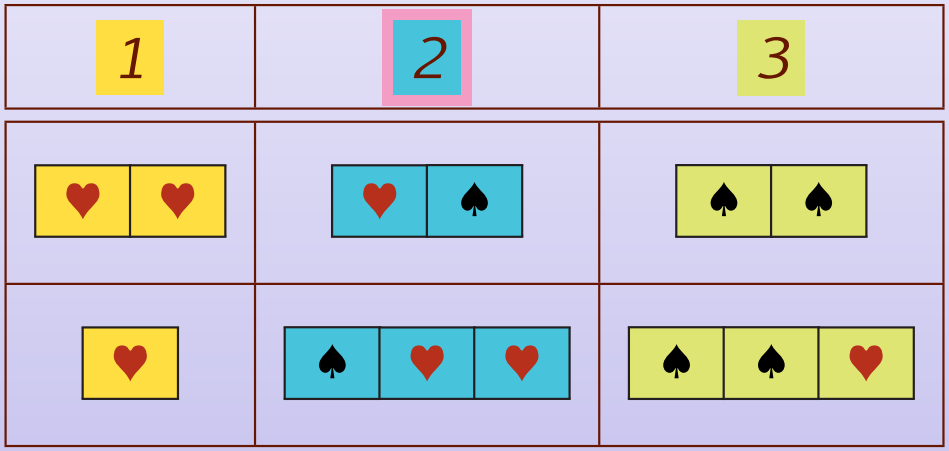

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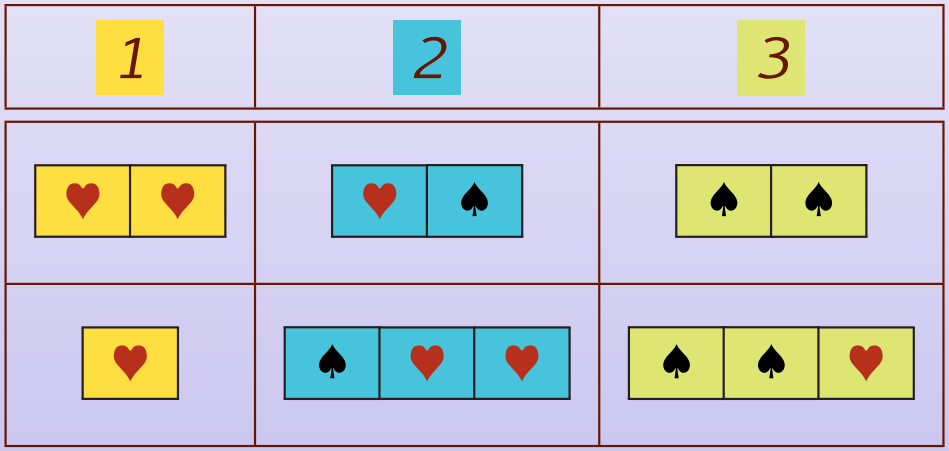

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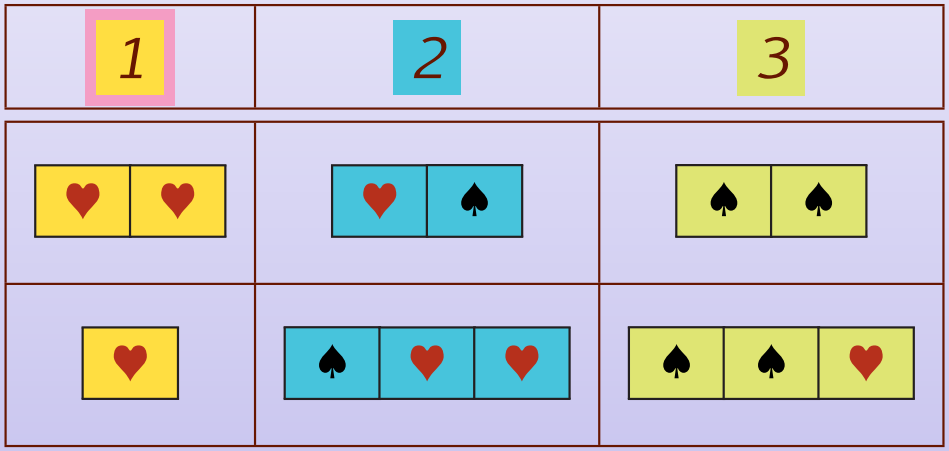

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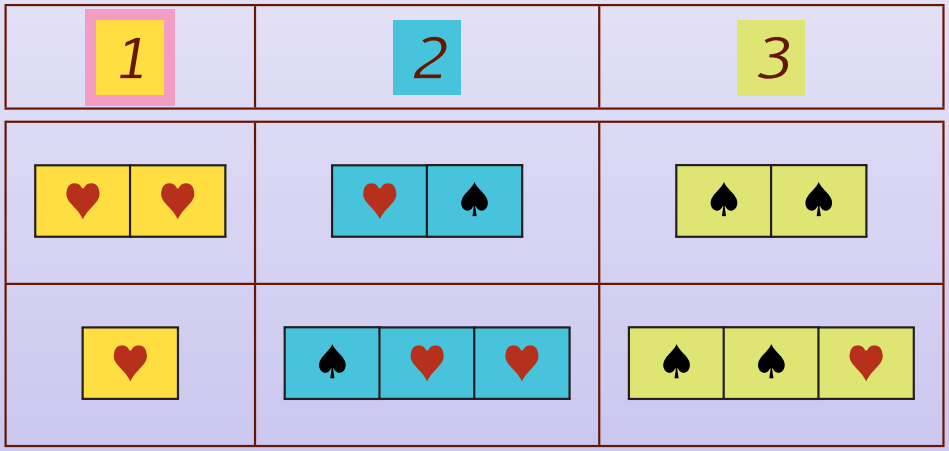

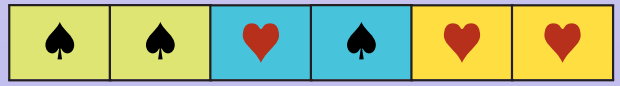

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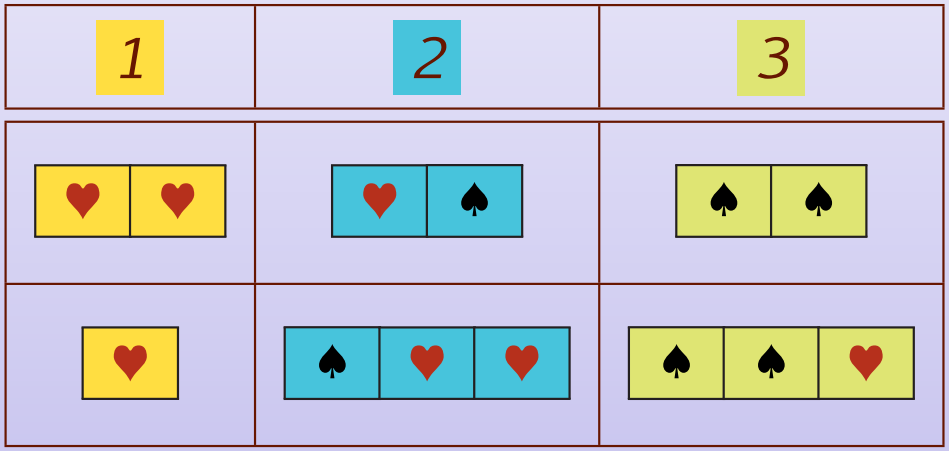

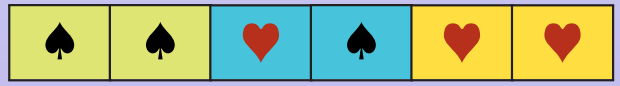

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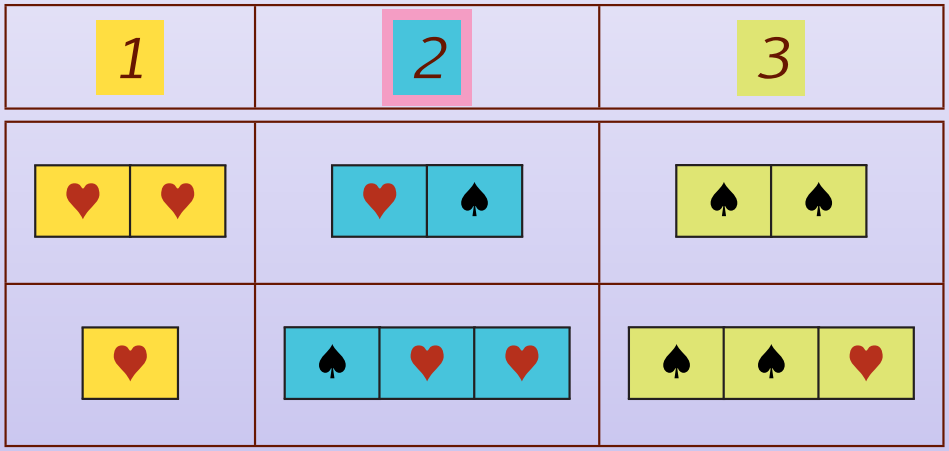

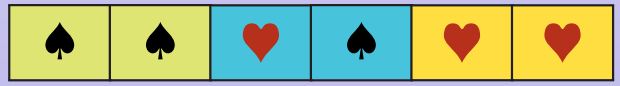

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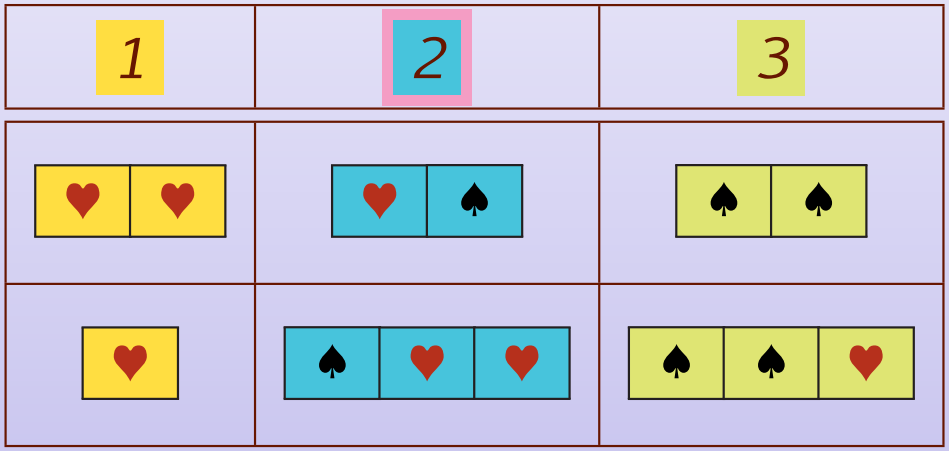

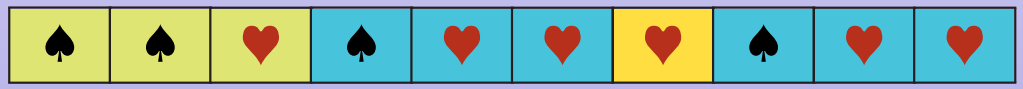

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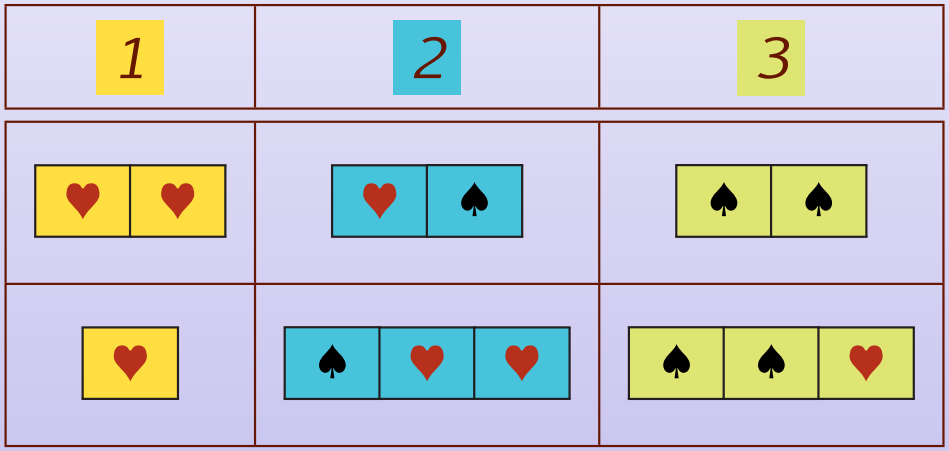

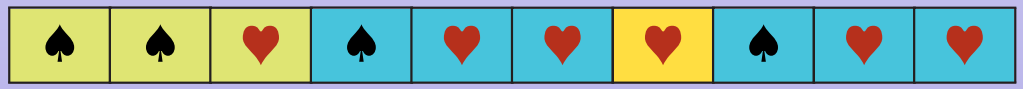

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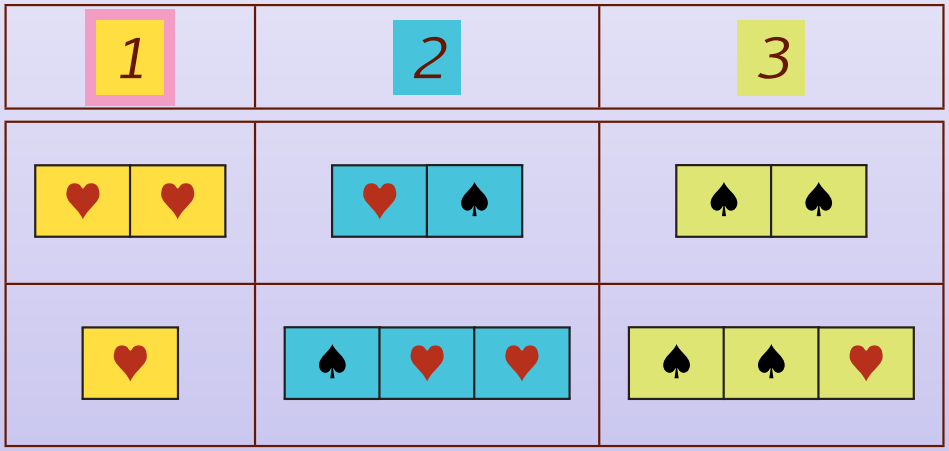

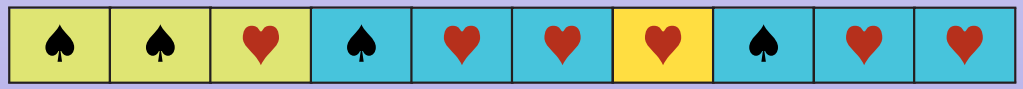

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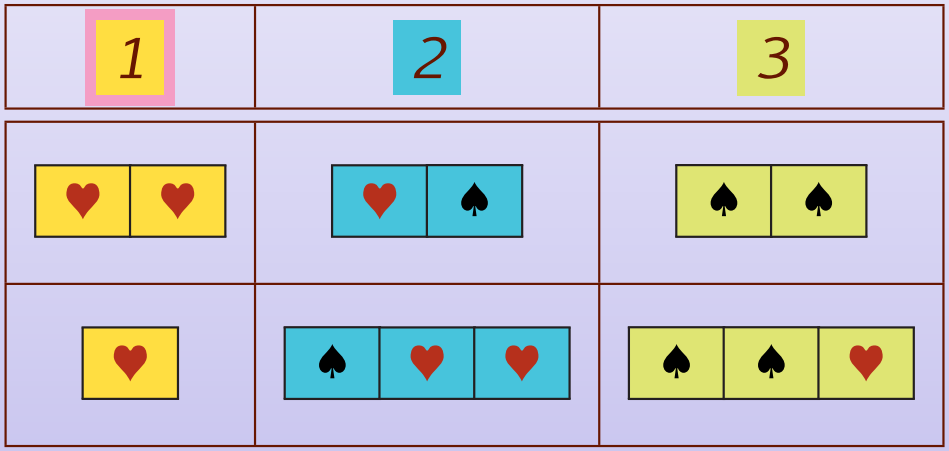

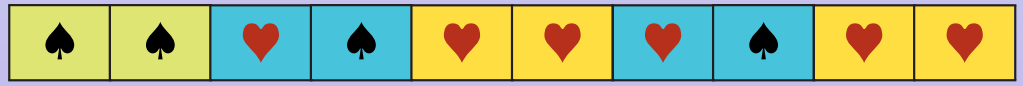

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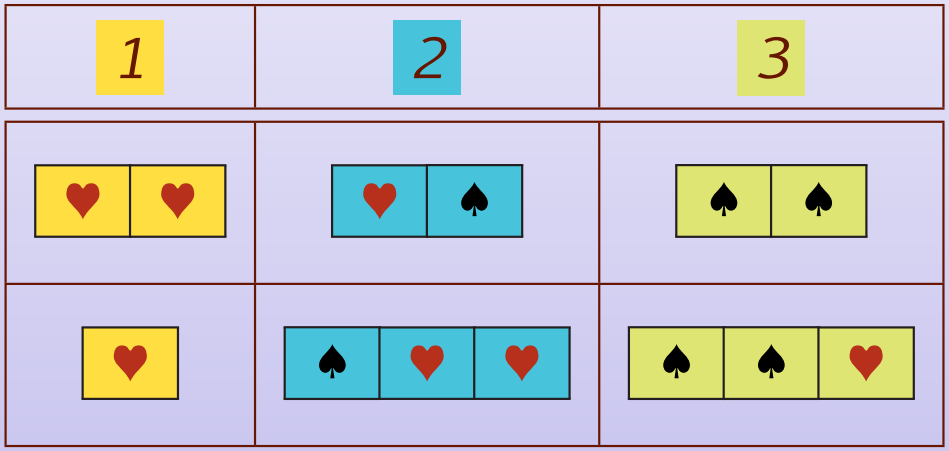

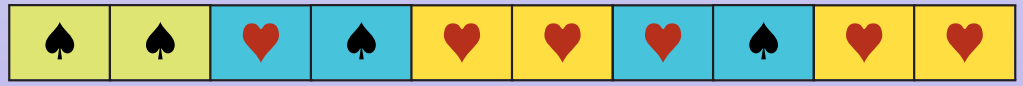

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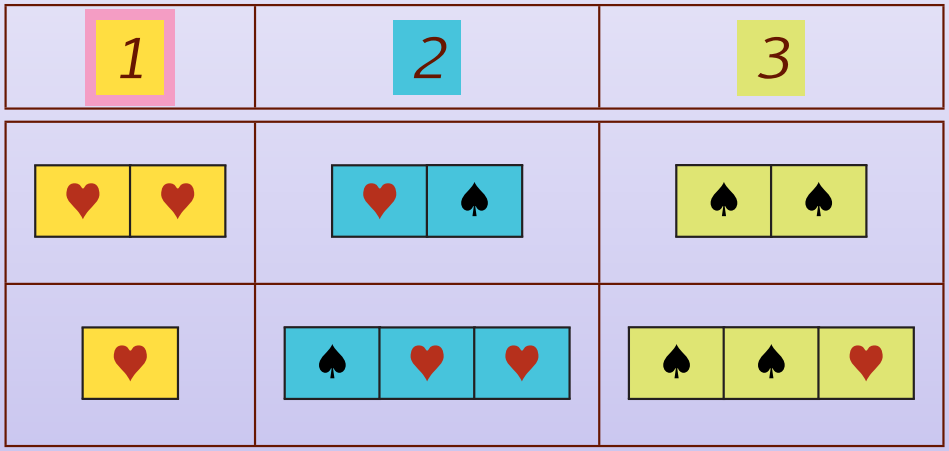

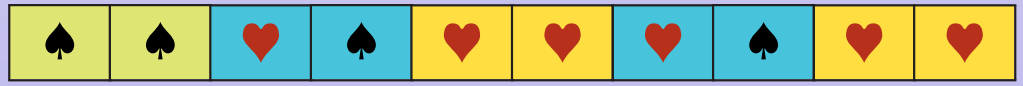

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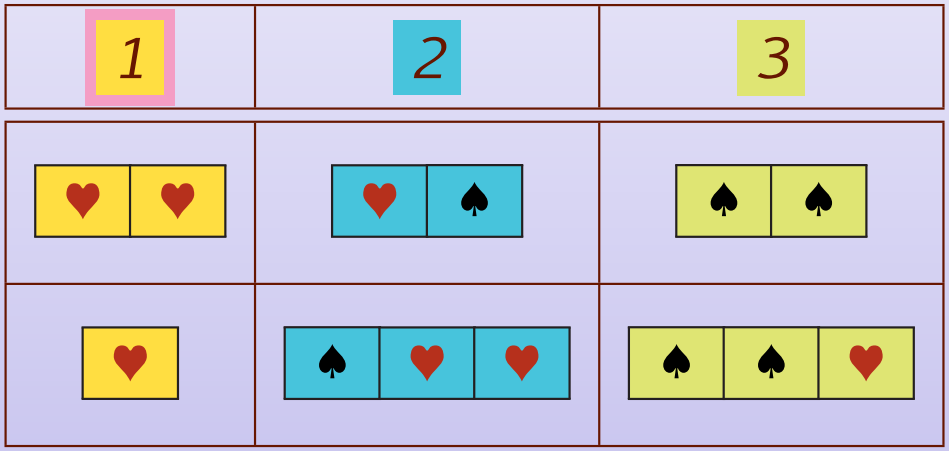

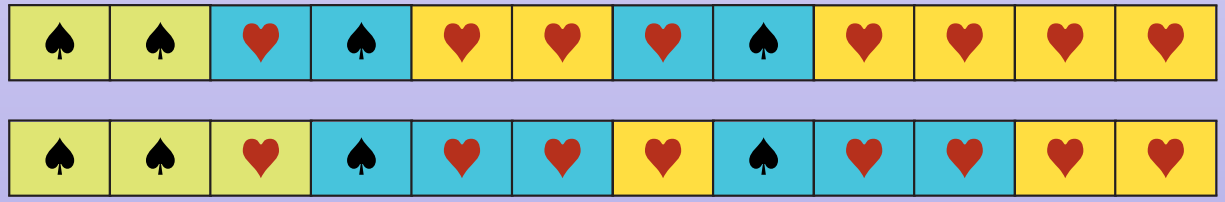

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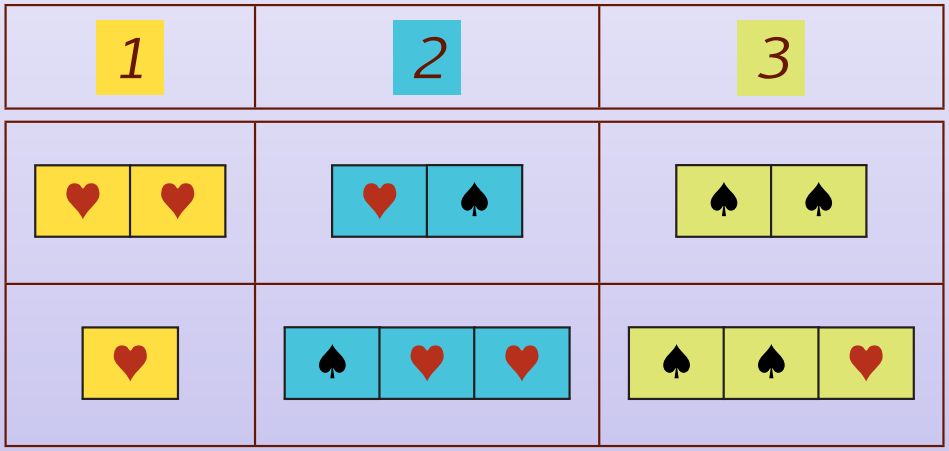

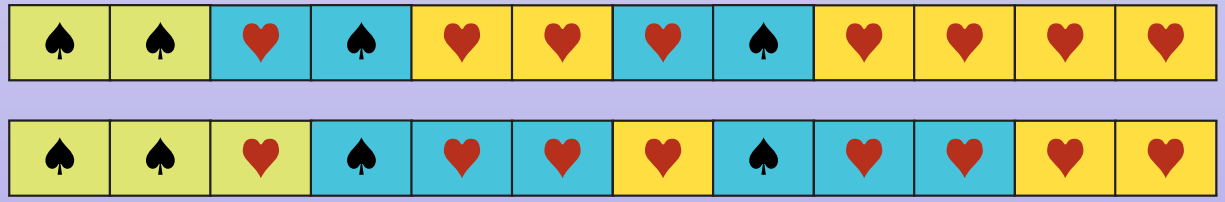

Done

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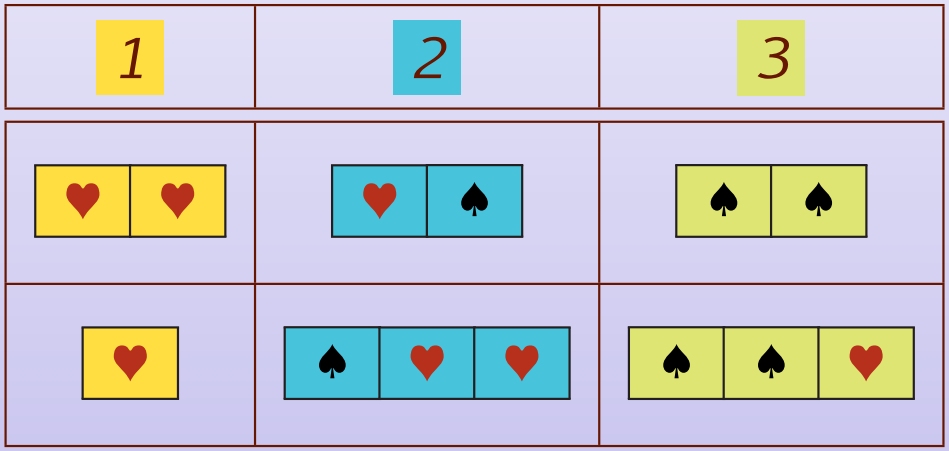

2nd play

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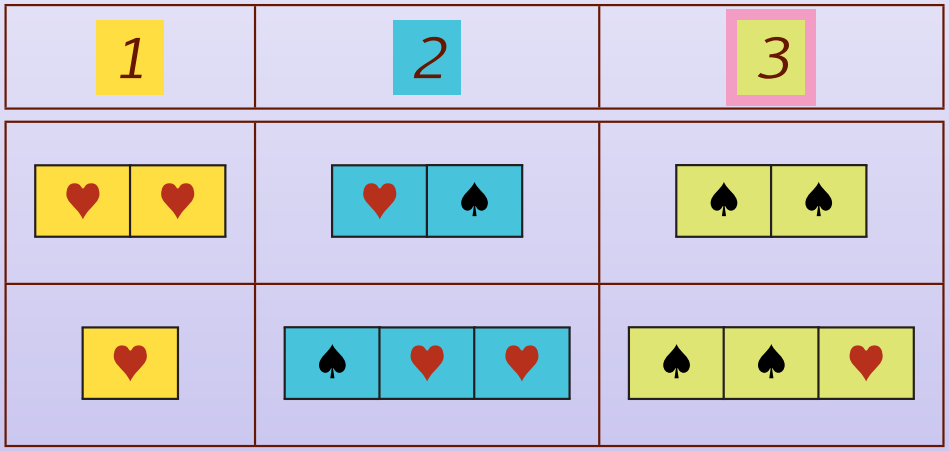

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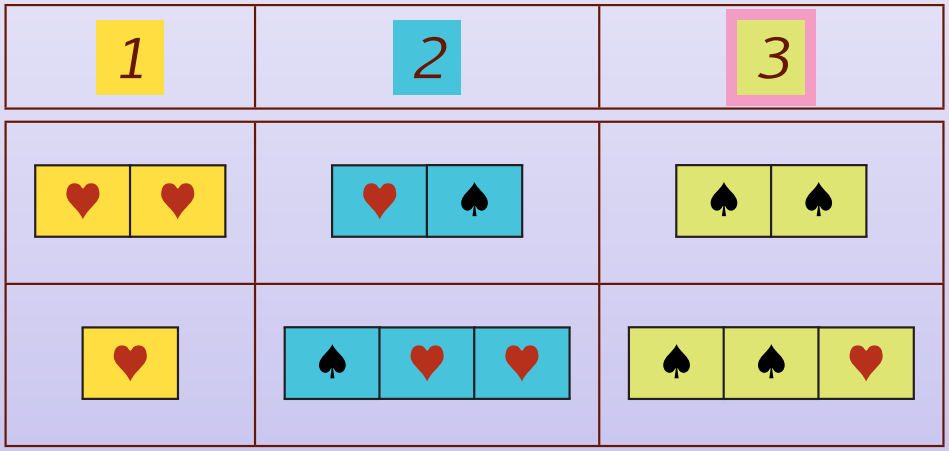

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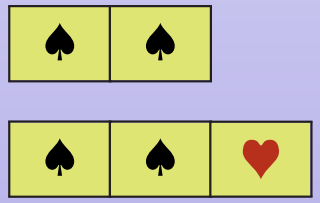

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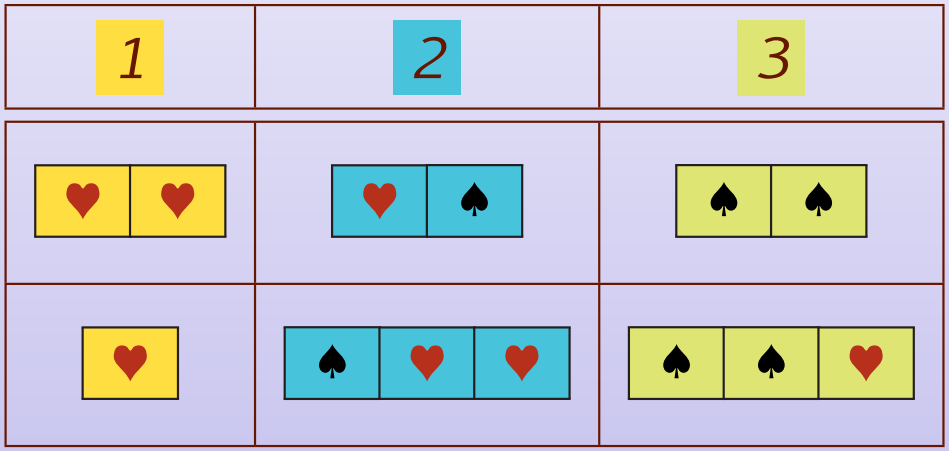

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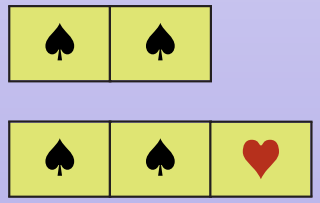

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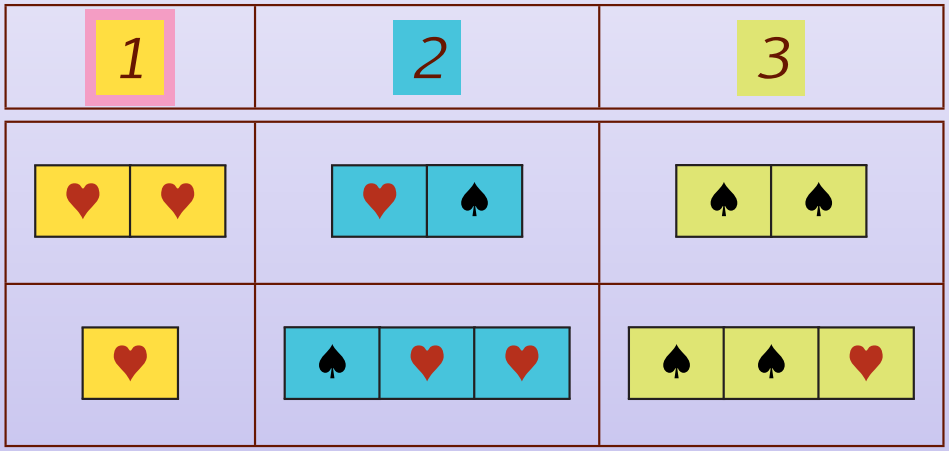

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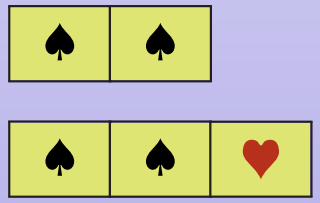

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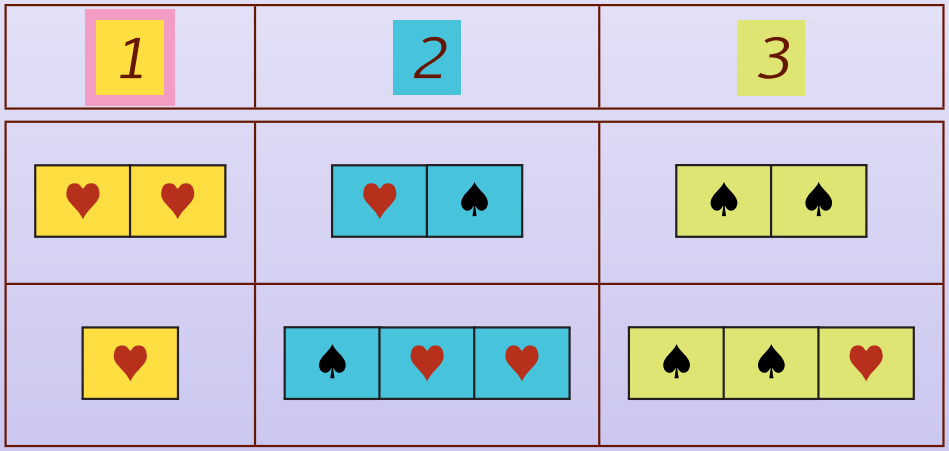

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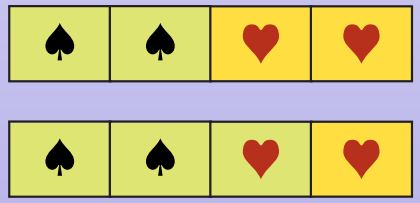

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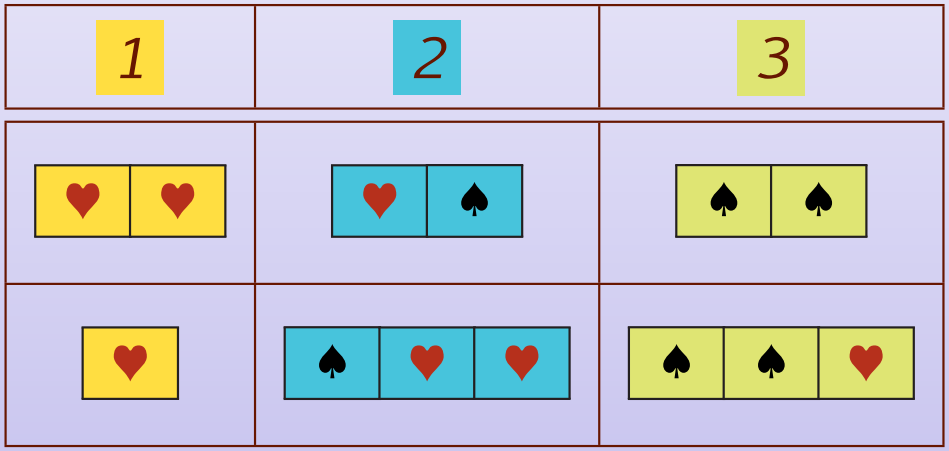

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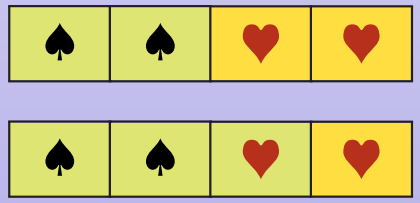

Done

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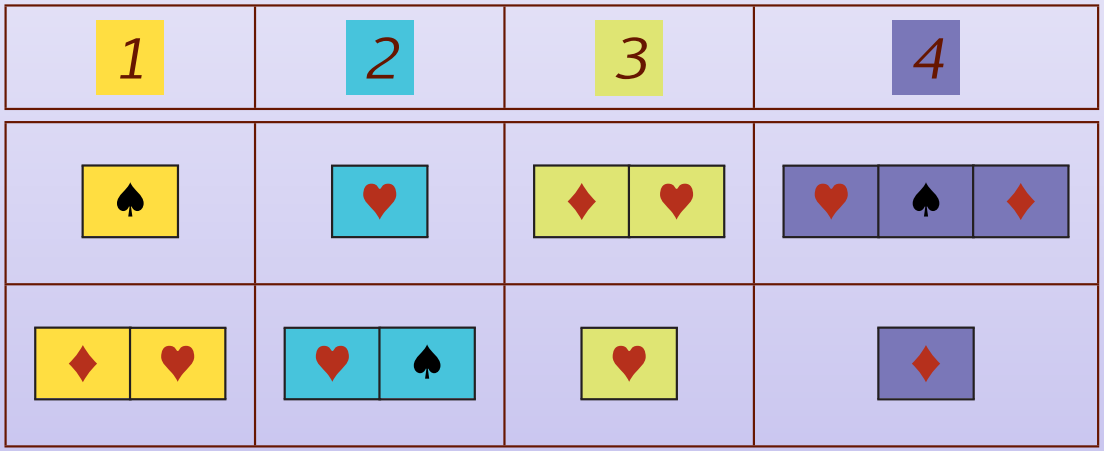

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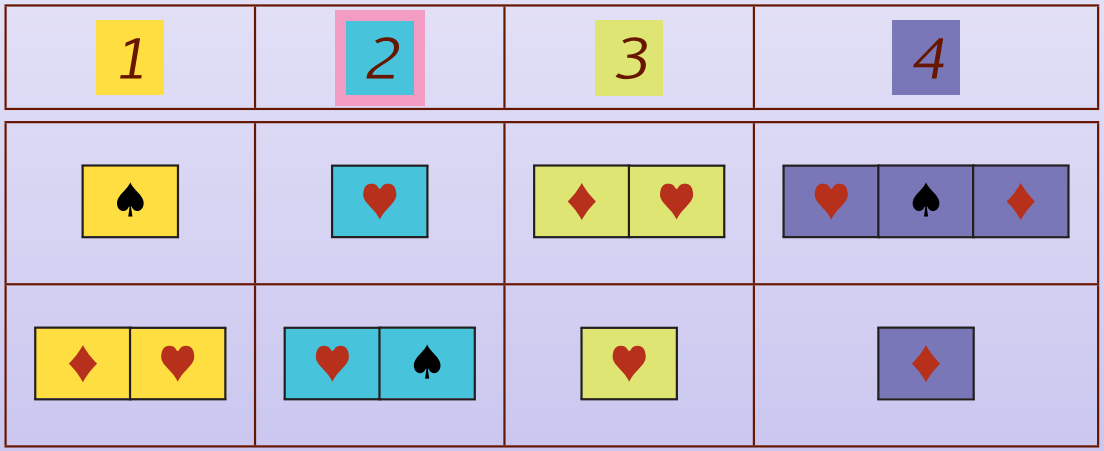

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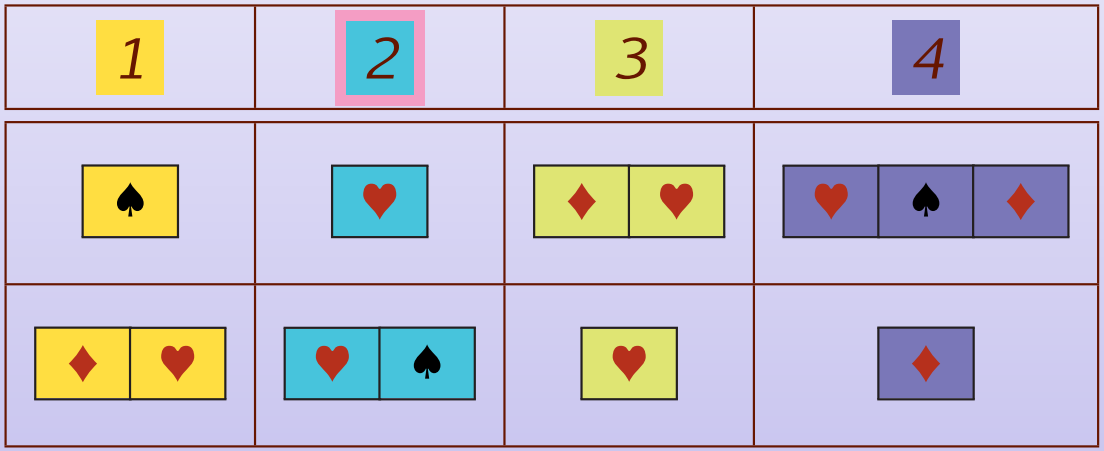

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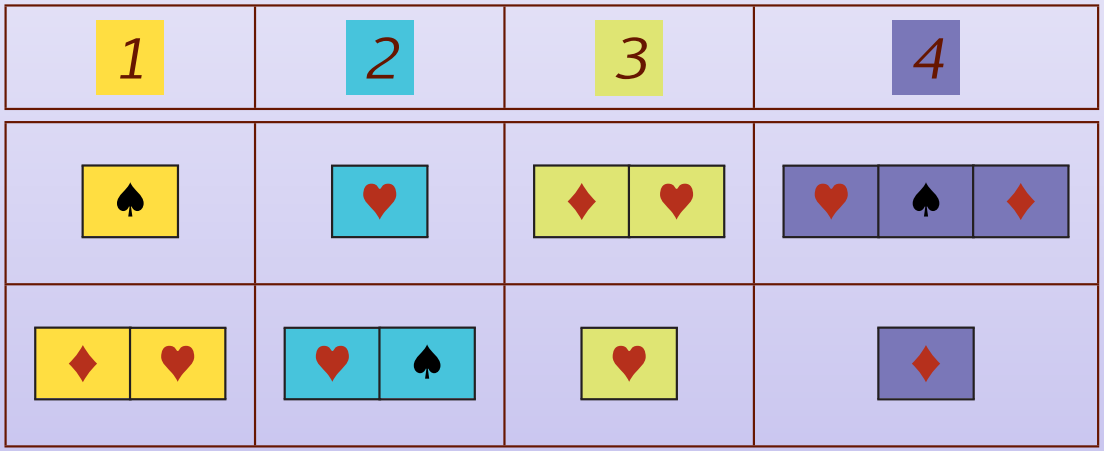

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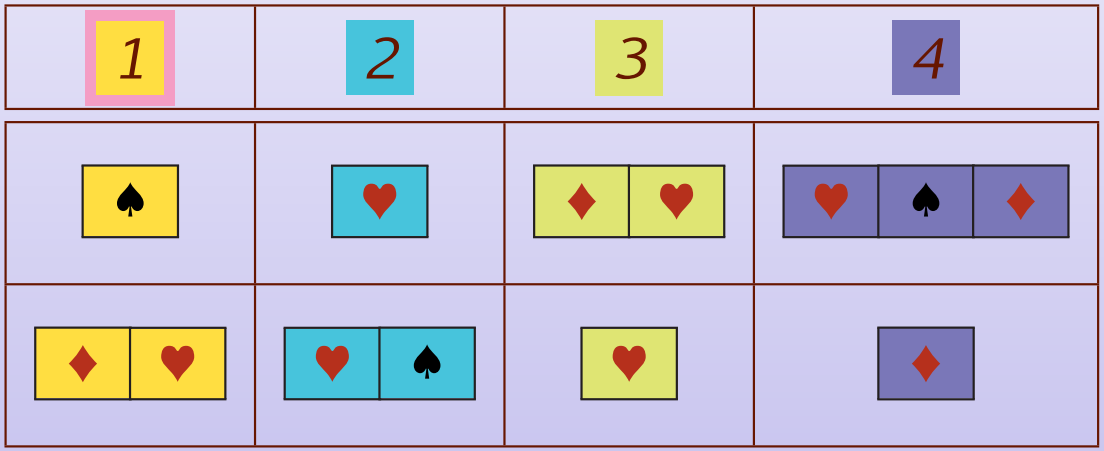

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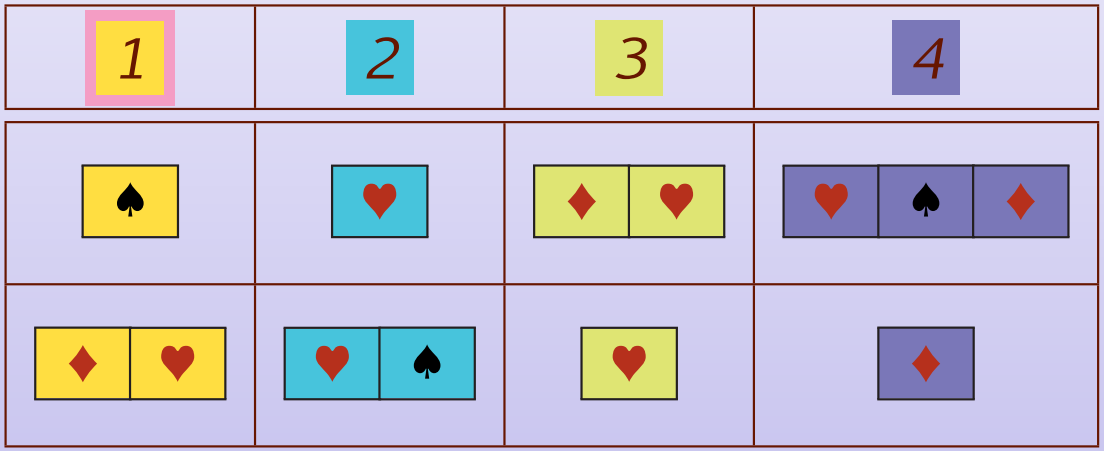

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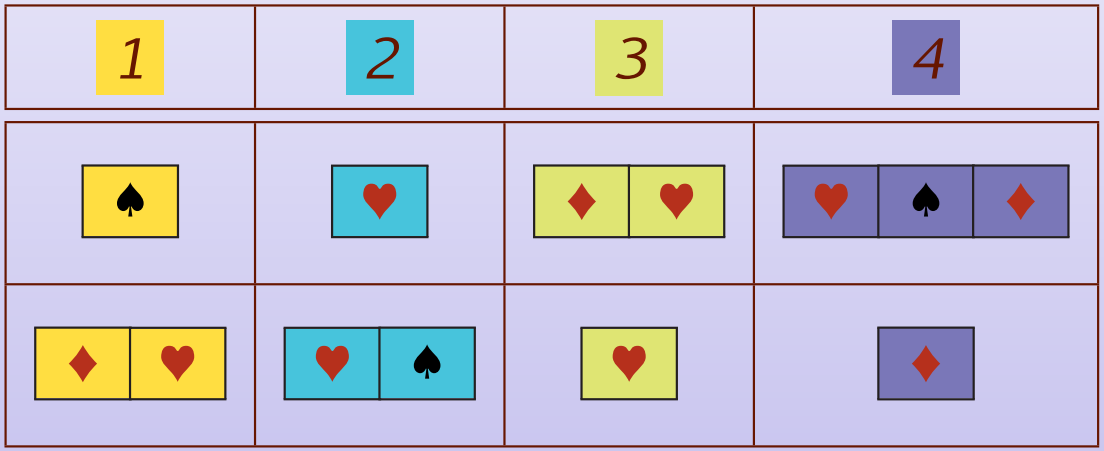

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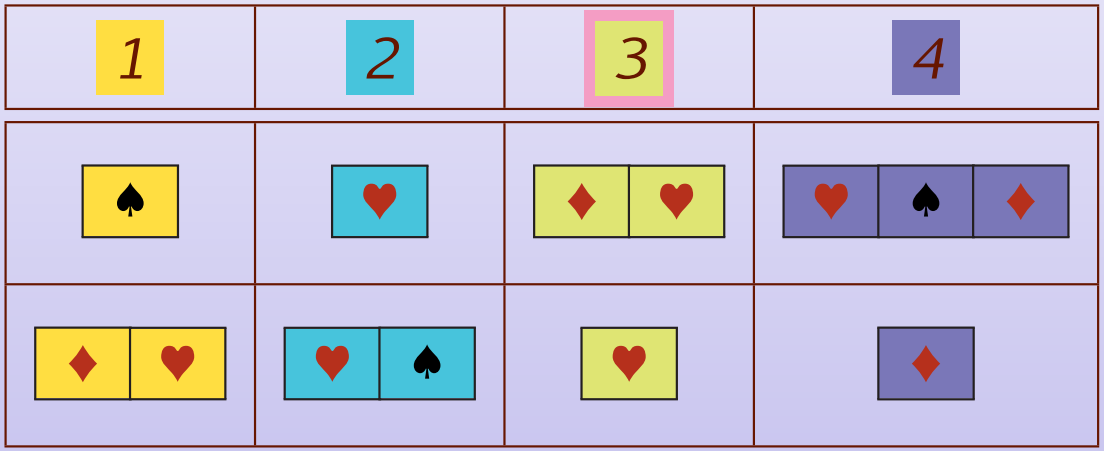

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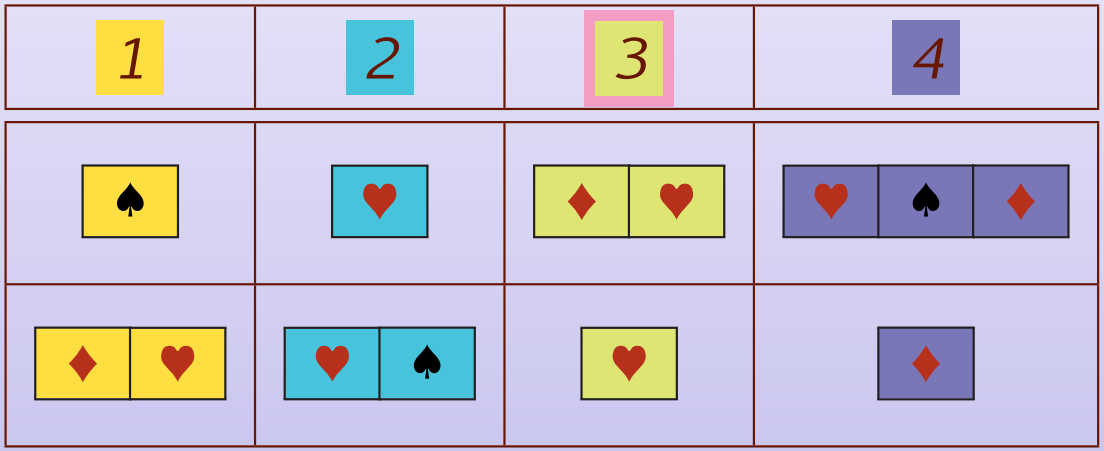

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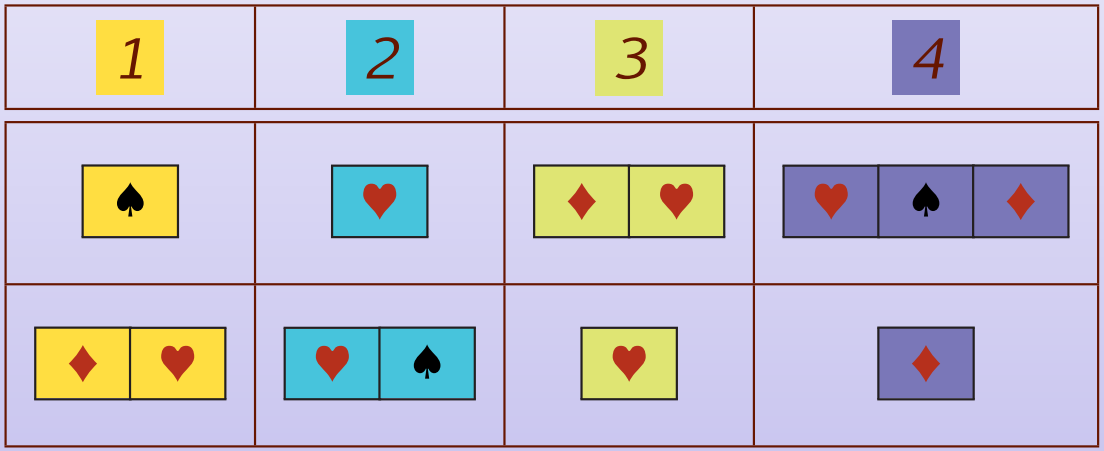

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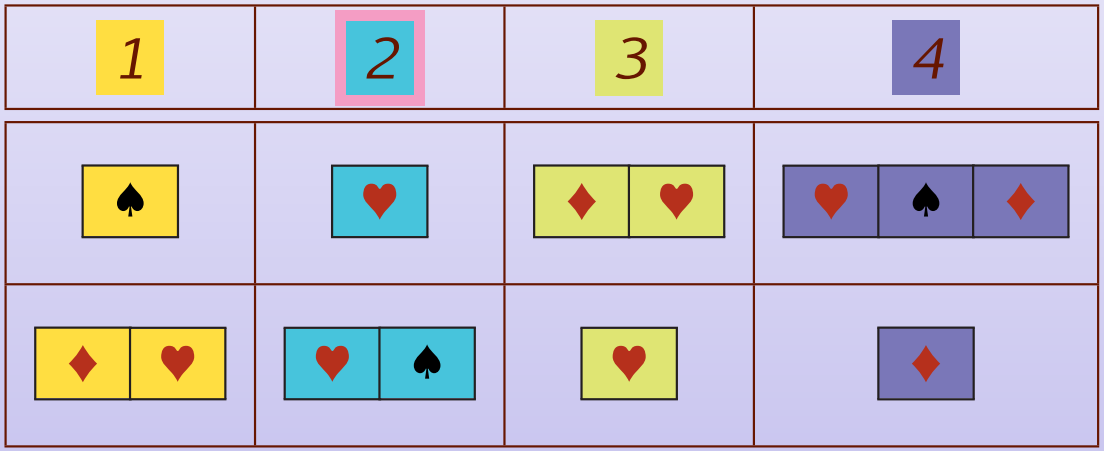

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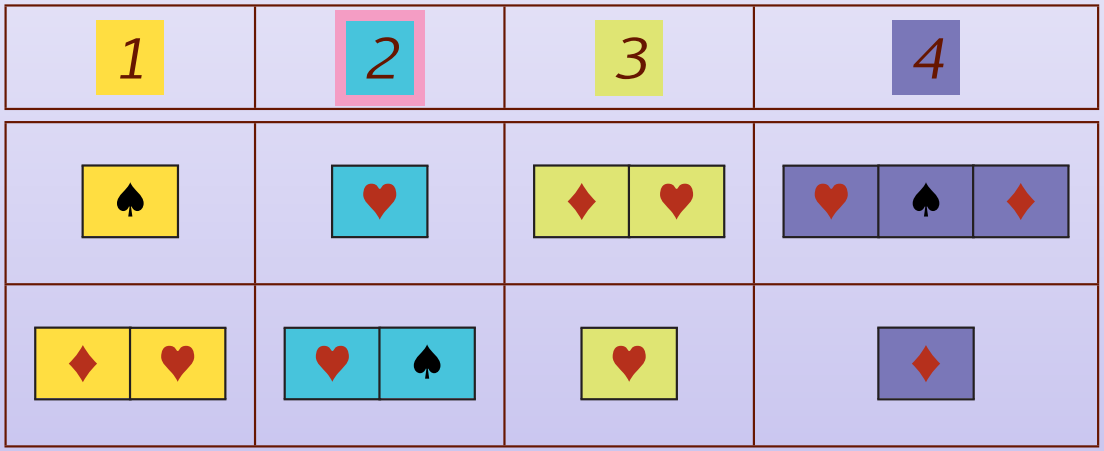

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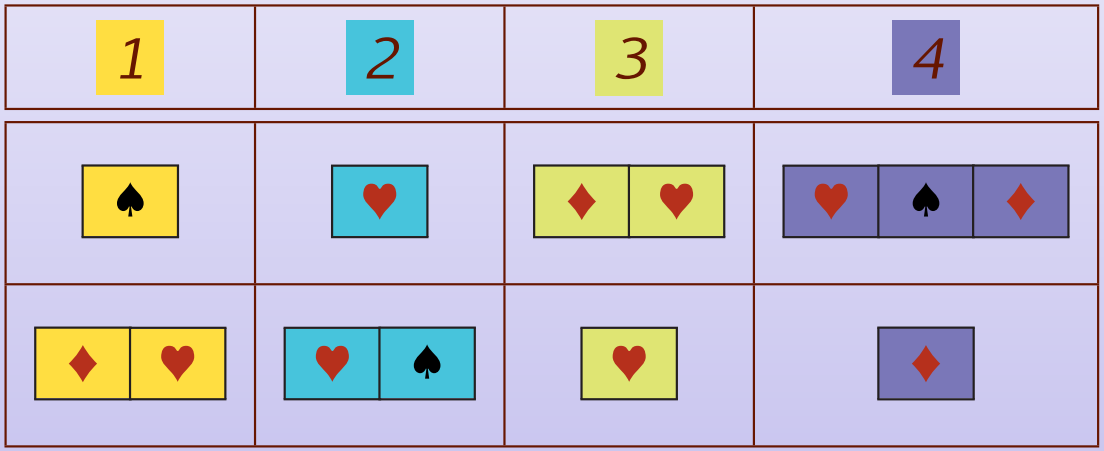

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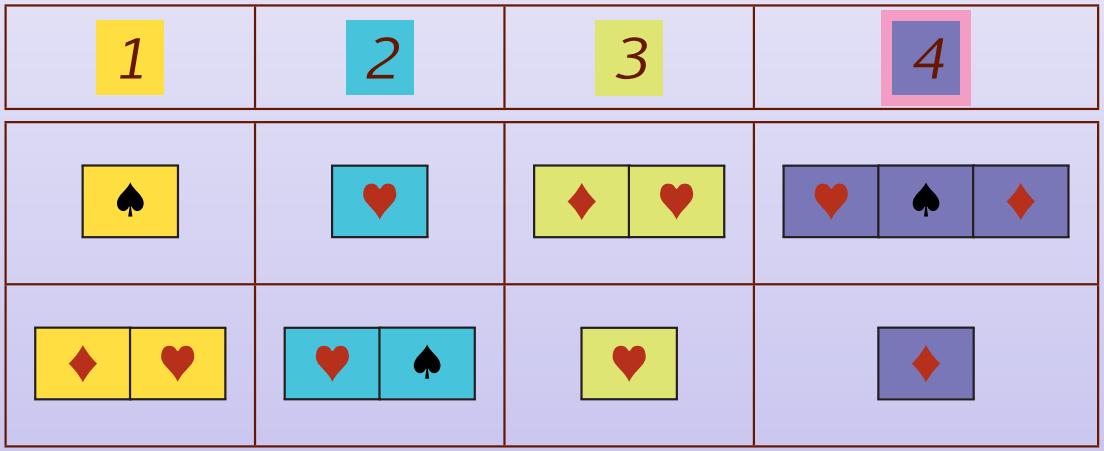

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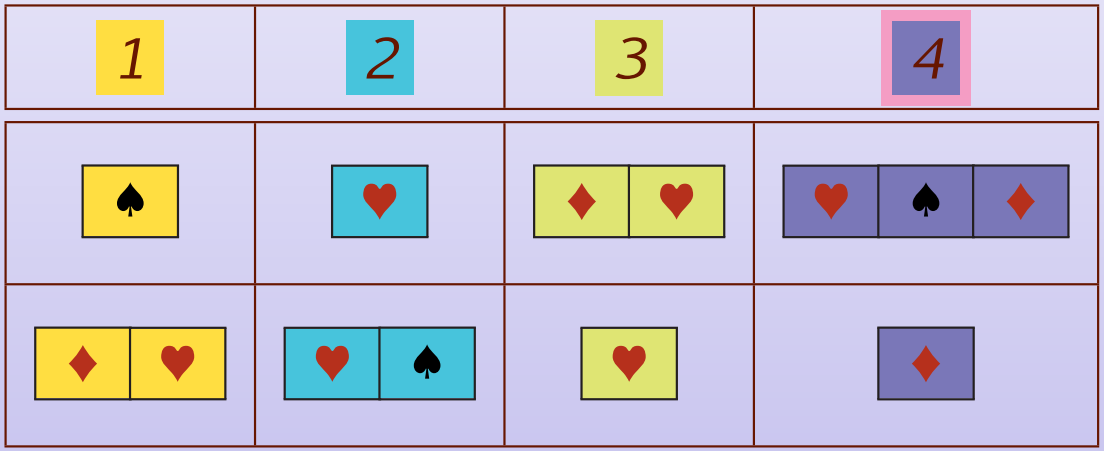

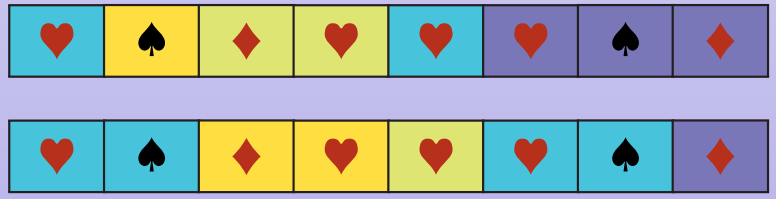

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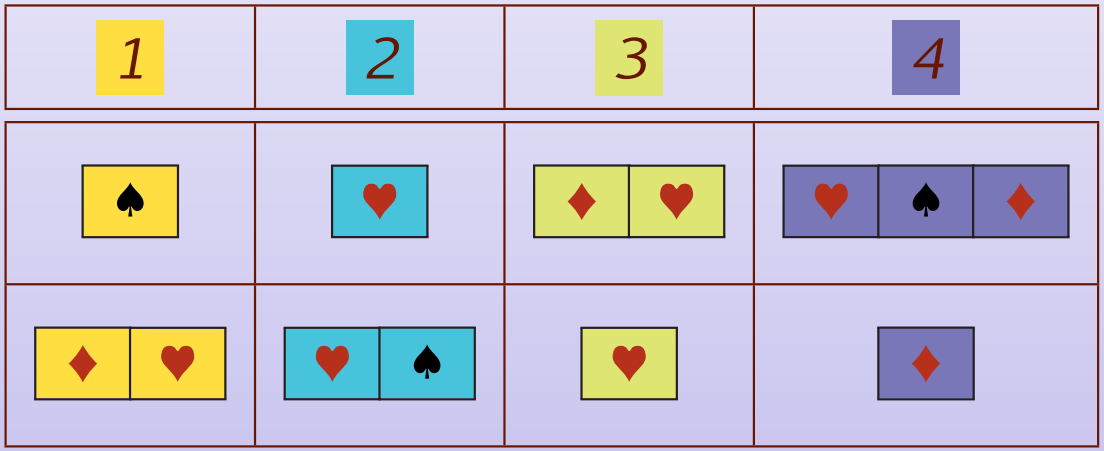

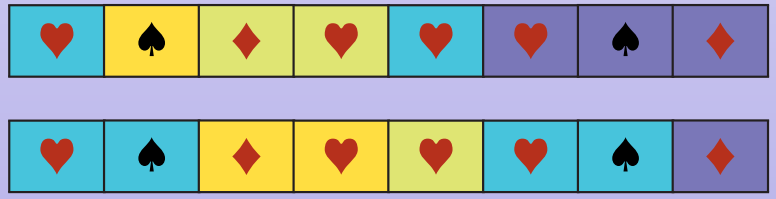

Done

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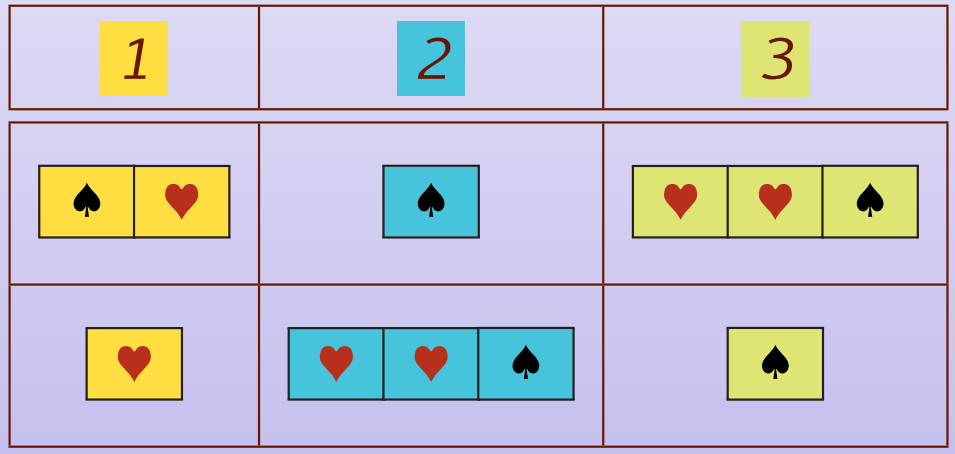

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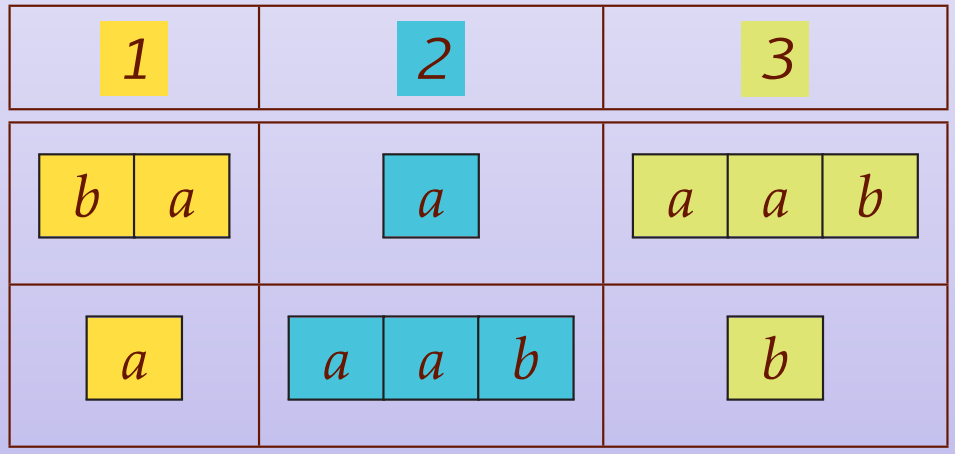

Last play

1	2	3
ba	a	aab
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We have a list of 3 pairs of words : $(ba,a),(a,aab),(aab,b)$.

In general, an instance is given be $n \in \mathbb{N}$ pairs of words.

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Write a program which takes as input a list of pairs of words and which prints

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Post's correspondence problem

Emil Post

1897 (Poland) – 1954 (USA)

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Problems for which no algorithm can exist are called

u n d e c i d a b l e

Decision problems

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Adjectives

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Example

Input : *a natural number* $n \in \mathbb{N}$
Question : *is* n *prime ?*

Evaluation or construction problems

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Input : *a natural number* $n \in \mathbb{N}$

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Also, any function is either computable or uncomputable

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Alonzo Church

1903 – 1995

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All feasible (unrestricted) models of computation lead to the same notion of computability.

No physical device can solve an undecidable problem or compute a function which is not computable.

Entscheidungsproblem (1928)

David Hilbert

1862 – 1943

Wilhelm Ackermann

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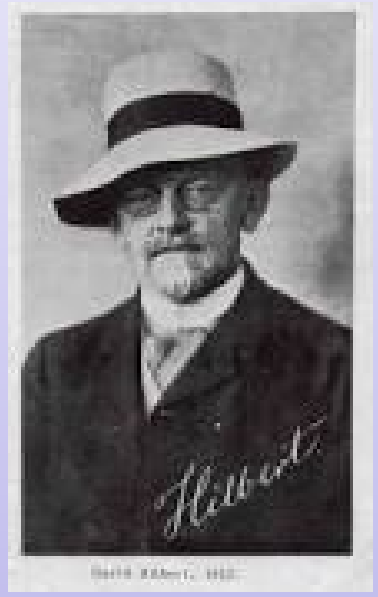

Wilhelm Ackermann

1896 – 1962

stated in 1928 «The Decision Problem» or «*Entscheidungsproblem*».

Input: a set of 1st order axioms Φ and a 1st order formula ψ.

Question: $\Phi \vDash \psi$?

The basic undecidable problem

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1912 – 1954

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Notation Let $\mathscr{P}$ be the set of all such programs.

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Remark

Either assertion

$$P(w) = \boxed{\text{yes}}$$

or

$$P(w) = \boxed{\text{no}}$$

implies that the running time of P on input w is finite.

The basic undecidable problem

The membership problem
(for recursively enumerable languages)

Input : a program $P \in \mathscr{P}$ and a string w
Question : $P(w) = \boxed{\text{yes}}$?

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Theorem (Turing 1936)
The membership problem undecidable.

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$$\begin{aligned} P_1 \text{ undecidable} &\Rightarrow P_2 \text{ undecidable} \\ P_2 \text{ decidable} &\Rightarrow P_1 \text{ decidable} \end{aligned}$$

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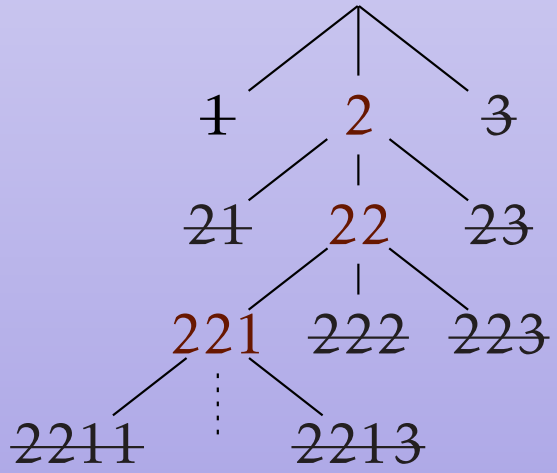
ba	a	aab
a	aab	b



Hilbert's tenth problem

Input: a multivariate polynomial $p(x_1,\ldots,x_n)$ with integer coefficients.

Question: Has

$$p(x_1,\ldots,x_n) = 0 \qquad \text{(a Diophantine equation)}$$

a solution in $\mathbb{Z}^n$?

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Theorem (Matiyasevich 1970)

Hilbert's tenth problem is undecidable.

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The following problem

Input : $\psi \in \Phi_1$

Question : *Does* ψ *hold in the field real numbers ?*

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Some consequences of Matiyasevich's Theorem

Theorem

The following problem

Input : *a system of ordinary differential equations*

$$\begin{cases} p_1(x, f_1(x), f_2(x), \ldots, f_n(x), f_1'(x)) = 0 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \quad \vdots \\ p_n(x, f_1(x), f_2(x), \ldots, f_n(x), f_n'(x)) = 0 \end{cases}$$

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Let $\mathscr{F}_4$ and $\mathscr{F}_5$ be classes of functions of one real variable s.t.

- $\mathscr{F}_2 \subseteq \mathscr{F}_4$
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In addition to undecidable problems, there are decidable problems which are untractable because they require an exponential amount of resources with respect to the input size.

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Example

Input: a finite set E, a set of mappings $\mathscr{F} \subseteq E^E$ and a map $f : E \to E$.

Question: Is f a composition of some mappings from $\mathscr{F}$?


Thank you
神威
太湖之光