Ben Elias Chennai Lectures Part I 1/2014 1

<u>This talk:</u> First, give rough intro to KL conjectures, motivation for the whole masterclass, will avoid reliance on rep. thry background. Goal of class: prove + understand Soergel Conj ($\Rightarrow$ KL Conj.)

In second half, get cracking with intro to Hecke algebras

§1 Projectives in $\mathcal{O}_0$

of a $\mathbb{C}$ fd. semisimple lie alg. $\rightsquigarrow W$ Weyl group

Ex: $\mathfrak{g} = \mathfrak{sl}_n$ $W = S_n$.

$\rightsquigarrow$ BGG category $\mathcal{O}_0$ $\leftarrow$ triv central char, if you know you know, an abelian category.

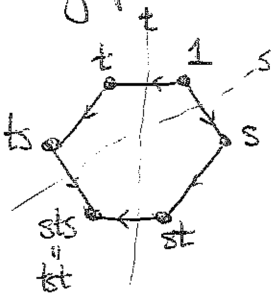
Facts: ① For each $w \in W$ have

$$\underset{\text{indec proj}}{P_w} \twoheadrightarrow \underset{\text{standard/Verma}}{\Delta_w} \twoheadrightarrow \underset{\text{simple}}{L_w}$$

The groth gp $[\mathcal{O}_0] \cong \mathbb{Z}[W]$ with 3 bases: $\{[L_w]\}$ $\{[\Delta_w]\}$ $\{[P_w]\}$

<u>Big Q:</u> What are the c.o.b. matrices? Another way: we know how big Δ_w is, so it is our favorite basis. How big is P_w?

Some graphical notation: encode $\mathbb{Z}[W]$ by placing integers on the dual Coxeter complex of W. ← more later



Ex: $W = S_3 = \langle s,t \mid s^2 = t^2 = 1, \ sts = tst \rangle$

$s = (12)$ X|

$t = (23)$ |X

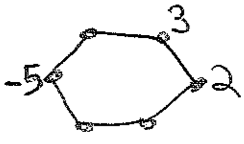
↑ simple reflections

Left mult by simple refl: reflection

Right mult: follow an edge

so label + orient edges

So $3 + 2s - 5ts$ is

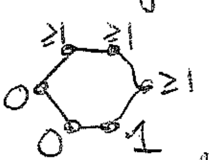


identify with $3[\Delta_1] + 2[\Delta_s] - 5[\Delta_{ts}]$ in $[\mathcal{O}_0]$

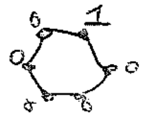
② A projective P_w has Δ-filt, i.e. a filt w/ subquots Δ_y.

Moreover, Δ_y appears $\Leftrightarrow y \leq w$, and Δ_w appears exactly once.

Ex: $[P_{st}]$ has form



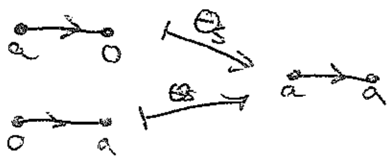
Ex: $P_1 = \Delta_1$

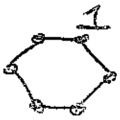


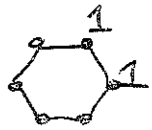
③ $\mathcal{O}_0$ has endofunctors Θ_s for each simple refl. which ① preserve projectives

~~wall-crossing~~ wall-bouncing functors ② act on groth gp by right mult by $1+s$

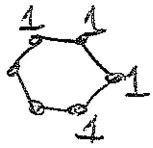
Key tool for exploring projectives.



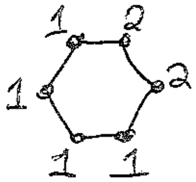
<u>Ex</u>: (for $\mathfrak{sl}_3$) we know P_1 

$\Theta_s P_1$ 

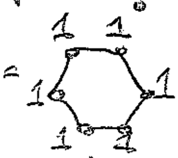
projective $\rightsquigarrow$ must be P_s ! (can't be $P_s \oplus P_1$ b/c then P_s is too small)

$\Theta_t \Theta_s P_1$ 

proj $\rightsquigarrow$ must be P_{st} !

$\Theta_s \Theta_t \Theta_s P_1$ 

proj ... is it P_{sts} ?

$P_{sts} \oplus P_s$? $\leftarrow$ correct answer, so $P_{sts} =$ 

$P_{sts} \oplus P_1$?

Which lower terms split off ?

but not always so easy !! (Verma's mistake)

$\Theta_s P_w = P_{ws} \oplus$ ambiguous lower terms.

④ For projectives P, Q, $\dim \mathrm{Hom}(P,Q) = ([P],[Q]) \leftarrow$ dot product, symmetric.

<u>Ex</u>: $\dim \mathrm{Hom}(P_s, \Theta_s\Theta_t\Theta_s P_1) = 4 = \dim \mathrm{Hom}(\Theta_s\Theta_t\Theta_s P_1, P_s)$

But somehow, inside this 4D space, is a 1D subspace giving the inclusion/projection of the summand.

How to find it ?? How to know what part of the 4D space splits off ?

KL conj designed to answer these Qns. But works with something non-geometric, non-RT, just algebraic.

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§2 | KL basis | ~~The Hecke alg~~ <u>Def</u>: A <u>Coxeter gp</u> is a gp with a presentation

$$W = \langle S \mid s^2 = 1,\ \underbrace{stst\ldots}_{m_{st}} = \underbrace{tsts\ldots}_{m_{st}} \rangle \qquad m_{st} \in \{2, 3, \ldots, \infty\}$$

simple reflections (S), quadratic ($s^2=1$), braid

braid, given quad, is equiv to $(st)^{m_{st}} = 1$

<u>Think</u>: s is reflection across hyperplane, st = rotation by 2θ so m_{st} records θ.

$\mathbb{Z}[W]$ then has presentation $\mathbb{Z}\langle s \rangle / (s+1)(s-1) = 0$, braid

<u>Def</u>: The Hecke alg of W ~~...~~ is a $\mathbb{Z}[v, v^{-1}]$ deformation of $\mathbb{Z}[W]$, ~~..~~ with presentation

$$\mathbb{H}_W = \mathbb{Z}[v^{\pm}]\langle H_s \rangle / (H_s + v)(H_s - v^{-1}) = 0$$

$$H_s H_t \ldots = H_t H_s \ldots$$

It has a basis $\{H_w\}$ where $H_w = H_{s_1}H_{s_2}\cdots H_{s_d}$ for any reduced expression $w = s_1 s_2 \cdots s_d$. H_1 = identity

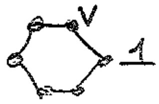
Ex: 

is $(1-v^2) + v^{-3}H_s + vH_{ts}$

Now $\mathbb{H}_W$ has an alg involution given by $v \mapsto v^{-1}$, $H_s \mapsto H_s^{-1}$ called the bar involution. $x \mapsto \overline{x}$.

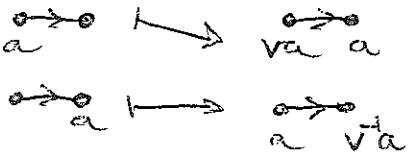
Exercise: $H_s^{-1} = H_s + (v - v^{-1})$

Exercise: $\underline{H}_s = H_s + v$ is bar-invariant, i.e. self-dual.



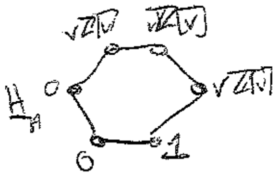
So mult by $\underline{H}_s$ preserves self-dual elements.

Exercise: right mult by $\underline{H}_s$ sends



Thm (K-L): $\exists!$ basis $\{\underline{H}_w\}$ of $\mathbb{H}_W$ s.t. ① $\underline{H}_w$ is self-dual
② $\underline{H}_w = H_w + \sum_{y<w} h_{yw} H_y$ with $h_{yw} \in v\mathbb{Z}[v]$.

Big Q: What are they?

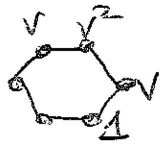


Uniqueness - exercise.

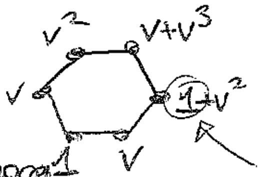
Construction: Clearly $\underline{H}_1 = H_1$ works. Now right mult by $\underline{H}_s$ to preserve self-duality

$1 \cdot \underline{H}_s$ 

$= \underline{H}_s$ ☑

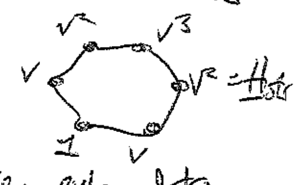
$1 \cdot \underline{H}_s \underline{H}_t$ 

$= \underline{H}_{st}$

$1 \cdot \underline{H}_s \underline{H}_t \underline{H}_s$ 

not $\underline{H}_{sts}$

can't subtract H_s, must subtract self-dual $\underline{H}_s$

$\underline{H}_s\underline{H}_t\underline{H}_s - \underline{H}_s =$ 

$= \underline{H}_{sts}$

$\underline{H}_w \underline{H}_s = \underline{H}_{ws} +$ lower terms ← ambiguous

This is basically the same as our computation of projectives in $\mathcal{O}_0$, except that the extra data (poly vs. integer) tells you exactly what to subtract!!!!! Don't know you get positive coefficients!!

Conj (K-L): $h_{yw}(1)$ = mult of Δ_y in P_w.

Moreover, $\mathbb{H}_W$ also has a pairing $(\ ,\ )$ "dot product" easy to compute for self-duals. Part I

$(\underline{H}_s\underline{H}_t\underline{H}_s, \underline{H}_s) = 1+2v^2+v^4$ ... really, this is because Hom spaces in $\mathcal{O}_0$ are secretly graded. Ind/Res is a degree 0 map, this "explains" how to find the right 1D space inside the 4D space. This is the easy part in the proof of the KL conjecture. ~~(We won't go into details on graded cat. $\mathcal{O}$, instead work with an obviously graded analog, Soergel bimodules.)~~

So $\exists$ some kind of graded version of $\mathcal{O}_0$, $\mathcal{O}_0^{\mathbb{Z}}$, with $[\mathcal{O}_0^{\mathbb{Z}}] \cong \mathbb{H}_W$

$\Theta_s \longleftrightarrow$ right mult by $\underline{H}_s$

$\operatorname{grdim} \operatorname{Hom}(P,Q) = ([P],[Q])$

KL conj: $[P_w] = \underline{H}_w$. Note that $(\underline{H}_w, \underline{H}_y) = \delta_{wy} + v\mathbb{Z}[v]$

Spose we know it for $y \le w$. $\underline{H}_w \underline{H}_s = \underline{H}_{ws} + \sum_{y<ws} \mu(w,s,y) \underline{H}_y$

$\Theta_s P_w = ?$

$\dim \operatorname{Hom}_0 (P_y, \Theta_s P_w) = \mu(w,s,y) = \dim \operatorname{Hom}_0 (\Theta_s P_w, P_y)$

i.e. intersection form on $\operatorname{Hom}(\Theta_s P_w, P_y)$

composing, one has a $\mu(w,s,y)$-square matrix in $\operatorname{End}_0(P_y) = \mathbb{C}$.

if this matrix is nondegenerate, can find $\mu(w,s,y)$ orthogonal summands $P_y \overset{\oplus}{\subset} \Theta_s P_w$.

Doing this for all $y < ws$, what remains will have the right size and it is P_{ws}; and $[P_{ws}] = \underline{H}_{ws}$.

So KL conj $\iff$ intersection forms are nondegenerate.

This is the hard part. Nothing numerical about it, must really investigate composition of morphisms!!! not just size of morphism spaces.

Now, I said we didn't need to know $\mathcal{O}_0$. This is because we will work instead with Soergel bimodules, an alternate version of $\mathcal{O}_0$ which has the following advantages:

① Obviously graded ② Easy to define ③ Works for all Coxeter groups, not just Weyl groups.

§3, Structures on the Hecke alg.

~~Rmk: Can define $\mathbb{H}_W$ in slightly more generality~~

Notation: Fix (W, S) Coxeter system.

$\underline{w} = (s_1, s_2, \ldots, s_d)$ is an expression, a word in S. $w = s_1 \ldots s_d$ is the corresponding element.

Abuse notation, write $\underline{w} = s_1 \ldots s_d$, the underline indicates the word itself matters.

An expression is reduced, or a rex, if d is minimal among all exp for w. $l(w) = d$. (length)

Rexes all related only by braid relations. This was why H_w was well-defined.

A subexpression is $(\underline{w}, \underline{e})$ $(\underline{e} \subset \underline{w})$ where $\underline{e} = e_1 e_2 \ldots e_d$, $e_i \in \{0,1\}$

it expresses $\underline{w}^{\underline{e}} = s_1^{e_1} s_2^{e_2} \ldots s_d^{e_d}$.

Bruhat order: $v \leq w \iff \exists\, \underline{e}, \underline{w}$ rex s.t. $\underline{w}^{\underline{e}} = v$. (indep of $\underline{e}$, actually)

$\implies l(v) \leq l(w)$.

Now in $\mathbb{H}_W$, write $H_{\underline{w}} = H_{s_1} \ldots H_{s_d}$ $(\equiv H_w)$ Indep of red exp, depends on non red exp

$\underline{H}_{\underline{w}} = \underline{H}_{s_1} \ldots \underline{H}_{s_d}$ ← more useful generators. depends, even for rex.

want to understand what $\underline{H}_{\underline{w}}$ is!

Def: Given $\underline{e} \subset \underline{w}$ its defect is given as follows. Think of $\underline{e}$ as giving a label U1 or D1, U0 or D0

Bruhat stroll: $e_i = 1$ go there
$e_i = 0$ don't go... but gaze longingly

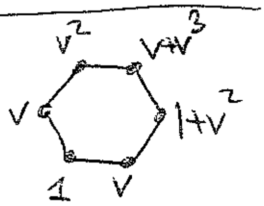
Also, discuss new presentation !!!!

Ex: $\underline{w} = sts$ $\quad \underline{e} = 100$ UUD $\quad \underline{e} = 001$ UUU

Defect $=$ $+1$ for each U0, -1 for each D0 $\quad d(\underline{e}) = 0 \quad d(\underline{e}) = 2$

Deodhar formula: $\underline{H}_{\underline{w}} = \sum_{\underline{e} \subset \underline{w}} v^{d(\underline{e})} H_{\underline{w}^{\underline{e}}}$

Ex: $\underline{H}_s \underline{H}_t \underline{H}_s =$



Finally $\mathbb{H}_W$ has 3 more structures, one alluded to:

$\omega : \mathbb{H}_W \to \mathbb{H}_W$ antiinvolution
$v \mapsto v^{-1}$
$\underline{H}_s \mapsto \underline{H}_s$
$\omega(ab) = \omega(b)\omega(a)$

$\varepsilon : \mathbb{H}_W \to \mathbb{Z}[v, v^{-1}]$
$H_w \mapsto \delta_{w1}$
Claim: $\varepsilon(ab) = \varepsilon(ba)$
std trace

$(,) : \mathbb{H}_W \times \mathbb{H}_W \to \mathbb{Z}[v, v^{-1}]$
$(a, b) = \varepsilon(\omega(a) b)$
Claim: $(H_x, H_w) = \delta_{wx}$
$(\underline{H}_x, \underline{H}_w) \in \delta_{wx} + v\mathbb{Z}[v]$
std pairing