Chennai Lectures
January 2014

Second problem sheet

Frobenius extensions

1. Let $H \subset G$ be an inclusion of finite groups. Then $\mathbb{R}[H] \subset \mathbb{R}[G]$ is a Frobenius extension. Compute the units and counits of adjunction. What is ∂^2? Can this extension be graded in an interesting way?

2. Look up the definition of a Frobenius algebra object in a monoidal category. Show that when $A \subset B$ is a Frobenius extension, then $B \otimes_A B$ is a Frobenius algebra object in the category of B-bimodules.

Chevalley's Theorem

3. Suppose that W is a finite group acting faithfully on a euclidean vector space V of dimension n. Let R be the coordinate ring of V, and R^W the invariant subring. Chevalley's theorem states that when W is generated by reflections, then R^W is generated by n algebraically independent homogeneous polynomials, known as a "basic set" of invariants. The basic set itself is not unique, but the multiset of degrees of the polynomials in the basic set is determined by the group W.

a) (Type A) Find a basic set for the symmetric group S_n acting on its standard n-dimensional representation. Recall that this action is generated by the reflections which flip x_i with x_j for two standard basis elements, and keep the rest of the basis fixed.

b) (Type B) Find a basic set for the signed symmetric group SS_n acting on its standard n-dimensional representation. Recall that this action is generated by the reflections above, as well as the reflection which sends x_i to $-x_i$ and keeps the rest of the basis fixed.

c) (Type D) Find a basic set for the even signed symmetric group ESS_n acting on its standard n-dimensional representation. Recall that this action is generated by the symmetric group and by the reflection which sends x_i to $-x_j$ and x_j to $-x_i$, and keeps the rest of the basis fixed.

4. Some "counterexamples" to the Chevalley theorem:

a) Find an example where W is not generated by reflections, and R^W is **not** a polynomial ring, i.e. it is not generated by algebraically independent elements.

b) Find an example where W is infinite, and R^W is a polynomial ring, but with $n-1$ generators rather than n.

5. Let us examine the set of degrees $\{d_i\}$.

a) Show that the trace of w on the symmetric tensor $S^k V$ is given by the coefficient of t^k in

$$\frac{1}{\det(1-tw)}.$$

b) Show that the dimension of the invariant subspace V^W is given by the trace of

$$\frac{1}{|W|}\sum_{w\in W} w.$$

c) By computing the dimension of each graded piece of R^W, show that

$$\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det(1-tw)}=\prod_{i=1}^{n}\frac{1}{1-t^{d_i}}.$$

d) Recall that an element of W is a reflection if all but one eigenvalue is 1, and the remaining eigenvalue is -1. Let N denote the number of reflections in W (also the number of positive roots). Show that $\prod d_i = |W|$ and $\sum(d_i-1)=N$.

e) Verify that the basic sets you found in Q3 have the correct degrees. What must the degrees of a finite dihedral group be?

Dihedral groups

6. (*) Suppose that W is a dihedral group, with $S=\{s,t\}$ and $m=m_{s,t}$. Instead of writing $a_{s,t}=-2\cos(\frac{\pi}{m})$, let us just write $a_{s,t}=-(q+q^{-1})$. After all, when $q=e^{\frac{\pi}{m_{s,t}}}$, the two formula agree. This will allow us to write formulae which work simultaneously for all dihedral groups, using quantum numbers.

a) Consider the quantum number

$$[n]=\frac{q^n-q^{-n}}{q-q^{-1}}=q^{n-1}+q^{n-3}+\ldots+q^{3-n}+q^{1-n}.$$

One has $[1]=1$ and $[0]=0$. Find a formula for $[2][n]$ in terms of quantum numbers. Does this remind you of any formulas in previous exercises?

b) The statement that q^2 is a primitive m-th root of unity is equivalent to what statement about quantum numbers? The statement that q is a primitive $2m$-th root of unity is equivalent to what statement about quantum numbers? What about when q is a primitive m-th root of unity for m odd? Compare $[m-k]$ and $[k]$. Compare $[m+k]$ and $[m-k]$.

c) Compute the matrix for the action of $(st)^k$ on the 2-dimensional space spanned by α_s and α_t, in terms of quantum numbers. When does (st) have finite order m? When $m=2k+1$ is the order of (st), what is $(st)^k(\alpha_s)$?

d) Assume that q is a primitive $2m$-th root of unity. The *positive roots* for the dihedral group are the elements in the W-orbit of the simple roots $\{\alpha_s,\alpha_t\}$, which have the form $a\alpha_s+b\alpha_t$ for $a,b\geq 0$. Find a simple enumeration of these roots as linear combinations of α_s and α_t.

7. (*) Continuing Q6: Now we investigate the invariant subring R^W, in the case when m is finite.

a) Find a formula for a quadratic polynomial $z\in R$ for which $s(z)=t(z)=z$.

b) (Extra credit): Show that ∂_s and ∂_t satisfy the braid relations.

c) Let $\mathbb{L}$ be the product of the positive roots. Show that $s(\mathbb{L})=t(\mathbb{L})=-\mathbb{L}$.

d) Assume $m\leq 3$ and let w_0 be the longest element. What is $\partial_{w_0}(\mathbb{L})$? Care to generalize?

e) Suppose that $m=2$. Find dual bases $\{a_i\}$ and $\{b_i\}$ for R over $R^{s,t}$, under the pairing $(f,g)\mapsto\partial_{w_0}(fg)$. Show that $\sum a_ib_i=\mathbb{L}$.

f) Find a polynomial Z of degree m which is invariant. In fact, $R^W=\mathbb{R}[z,Z]$. Hint: Look at the roots of the dihedral group with twice the size.

8. Continuing Q7: (These exercises are more computational.)

a) Show that $R^{s,t} = \mathbb{R}[z, Z]$.

b) Suppose that $m = 3$. Find dual bases for R over $R^{s,t}$. Show that $\sum a_i b_i = \mathbb{L}$.

c) Suppose that $m = 3$. Find dual bases for R^s over $R^{s,t}$, under the pairing using $\partial_s \partial_t$. Show that $\sum a_i b_i = \frac{\mathbb{L}}{\alpha_s}$.

Generalizing the reflection representation

9. We will now use the term "Cartan matrix" to refer to any matrix indexed by S, satisfying $a_{s,s} = 2$ and $a_{s,t} = 0 \iff a_{t,s} = 0$, with coefficients in a base ring $\Bbbk$ (not necessarily integers). A Cartan matrix need not be symmetric, or even symmetrizable (i.e. conjugate by a diagonal matrix to a symmetric matrix).

a) Given a Cartan matrix, one can still construct a vector space $\mathfrak{h}^*$ with involutions $s \in S$ acting upon it. Show that (st) has order m if and only if $a_{s,t}a_{t,s}$ is algebraically equivalent to $[2]^2$ for q a primitive $2m$-th root of unity.

b) Show that any Cartan matrix admitting a representation of a Weyl group, and satisfying $a_{s,t} = 0 \iff a_{t,s} = 0$, is symmetrizable.

c) Show that the following matrix admits a representation of the affine Weyl group $\tilde{A}_4$, for any $q \in \mathbb{C}^*$. When is it symmetrizable? When is it conjugate, by a diagonal matrix, to a representation defined over $\mathbb{R}$?

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & -q^{-1} \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -q \\ -q & 0 & 0 & -q^{-1} & 2 \end{pmatrix}.$$

Bott-Samelson bimodules

10. (*) Exercises from class:

a) Verify all the statements made in lecture about the Demazure operator ∂_s.

b) Why is $BS(\underline{w})$ free as a right R-module?

c) Verify that $fc_s = c_s f$.

d) (Hardest) When $m_{st} = 3$, verify that $B_s B_t B_s \cong B_s \oplus (R \otimes_{R^{s,t}} R(3))$.

11. Construct a map $B_s \otimes_R B_t \to B_{s,t}$ sending $1 \otimes 1 \otimes 1 \mapsto 1 \otimes 1$, when $m = 2$. Why is there no such map when $m > 2$?

12. (*) Practice with the Soergel Hom formula:

a) Compute the size of the Hom space $\mathrm{Hom}(B_s, B_t)$. Find a generating set of morphisms, and indicate how these morphisms factor.

b) Compute the size of the Hom space $\mathrm{Hom}(B_s, B_s B_t B_s)$, assuming $m_{st} > 2$. Find a generating set of morphisms, and indicate how these morphisms factor.

c) Compose a morphism of minimal degree $B_s \to B_s B_s$ with one of minimal degree $B_s B_s \to B_s$. What is the resulting map? Show this by a general argument, and then by direct computation.