Chennai Lectures
January 2014

First problem sheet

Coxeter groups:

1. To practice with Coxeter groups, we play with some embeddings and foldings.

a) Let $\{s, t, u\}$ be the simple reflections inside the Coxeter group of type A_3. Show that the subgroup generated by (su) and t is a Coxeter group of type $B_2 = I_2(4)$, with simple reflections $\{su, t\}$, by checking the braid relation. This implies that B_2 embeds inside A_3 as the invariants under a certain automorphism σ, induced by a diagram automorphism.

b) Let $\{s, t, u, v\}$ be the simple reflections inside the Coxeter group of type A_4. Show that the subgroup generated by (su) and (tv) is a Coxeter group of type $H_2 = I_2(5)$, with simple reflections $\{su, tv\}$. However, this subgroup is not the invariants of any diagram automorphism.

c) Embed the Coxeter group of type $I_2(m)$ inside the Coxeter group of type A_{m-1} for $m \geq 3$, using products of distinct simple reflections.

d) Embed H_3 inside D_6. Embed H_4 inside E_8. Generalize this.

e) Look at star-shaped Coxeter groups: A_2, A_3, D_4, $\tilde{D}_4$, and so forth. Consider the subgroup generated by the hub and by the product of the spokes. What subgroups do you get?

2. Now we do the previous exercise "in reverse." Let (W, S) be a Coxeter group, and fix $s \in S$. Consider the set Γ_s of elements of W which have a unique reduced expression, and which have s in their right descent set. Γ_s has the structure of a labeled graph, where each element $w \in \Gamma_s$ is labeled by the (unique!) element $t \in S$ in its left descent set, and where w, v are connected by an edge if and only if $w = uv$ for some $u \in S$.

a) Let $\{s, t\}$ be the simple reflections in type B_2. Compute that Γ_s is A_3, with the labelings corresponding to the embedding of B_2 inside A_3 from Q1.

b) Do the same for $I_2(m)$ and A_{m-1}.

c) Let (W, S) be the Coxeter group of type H_4. For $s \in S$, compute the labeled graph Γ_s.

d) Repeat the exercise for $I_2(\infty)$. What labeled graph do you obtain?

(If you know about such things, Γ_s is the W-graph of the left cell containing s. See Lusztig "Some examples of square integrable functions on a p-adic group".)

3. Coxeter systems (W, S) are equipped with a standard length function ℓ, but can also be equipped with non-standard length functions, sometimes called *weights*. A weight L is a map $W \to \mathbb{Z}$ satisfying $L(uv) = L(u) + L(v)$ whenever $\ell(uv) = \ell(u) + \ell(v)$. Deduce the following elementary facts.

a) A weight function L is determined by the weights $L(s)$ of the simple reflections. Moreover, $L(s) = L(t)$ whenever m_{st} is odd.

b) Suppose one has an embedding of Coxeter groups $\iota\colon (W, S) \hookrightarrow (W', S')$ as in Q1, where each simple reflection $s \in S$ is sent to a product Πt of commuting simple reflections $t \in S'$. This equips (W, S) with a weight L, given by $L(s) = \ell(\iota(s))$. For each possible value of m_{st}, what are the possible values of the ratio of $L(s)$ to $L(t)$? It will help to remind oneself of the classification of finite Coxeter groups.

4. (*) Given an element $w \in W$, its *rex graph* $\tilde{\Gamma}$ is a graph constructed as follows:

- Vertices are reduced expressions for w.
- An edge connects two rexes if they differ by a single application of a braid relation. Label the edge with the number m_{st} associated to this braid relation.

Now, draw the following rex graphs.

a) Every element in type A_3 (most of them are uninteresting).

b) The longest element of every finite rank 3 Coxeter group: A_3, B_3, H_3 (the hardest), $A_1 \times I_2(m)$, $A_1 \times A_1 \times A_1$.

5. Let (W, S) have rank n. The *Coxeter complex* is a simplicial complex constructed as follows:

- There is an $(n-1)$-simplex labeled by w for each $w \in W$. The n faces of this $(n-1)$-simplex are labeled by the simple reflections s.
- Whenever $w = sv$, glue the simplices w and v along the face s. (Technically, one should fix the orientations when gluing faces. If $\ell(w) = \ell(v) + 1$ then glue the outward face of s in v to the inward face of s in w.)

Now, draw the Coxeter complex for the following Coxeter groups: $I_2(m)$ for m finite, $I_2(\infty)$, A_3 (the barycentric subdivision of a tetrahedron), B_3, $\tilde{A}_2$, $\tilde{B}_2$.

6. Continuing Q5: The *dual Coxeter complex* is a CW complex obtained by dualizing the Coxeter complex. In other words, there is a 0-cell for each simplex, a 1-cell connecting 0-cells if the simplices meet in a (codimension 1) face, a 2-cell glued along 1-cells if the faces all meet in a codimension 2 face, etc. Show that the dual Coxeter complex can be constructed directly as follows:

- There is a 0-cell for each $w \in W$. Said another way, there is a 0-cell for each coset of the trivial subgroup.
- There is a 1-cell for each pair $\{w, ws\}$ with s a simple reflection. Said another way, there is a 1-cell for each coset of each rank 1 parabolic subgroup.
- There is a 2-cell for each coset of each **finite** rank 2 parabolic subgroup.
- ...
- There is a k-cell for each coset of each finite rank k parabolic subgroup.
- However, if the rank of W is n, then the process ends at $k = n - 1$.

Also, draw the dual Coxeter complex for the same list of Coxeter groups.

Remark. In fact, one can also construct the *completed dual Coxeter complex* by also including the step $k = n$. This makes no difference when W is infinite, but glues in a single n-cell when W is finite. The resulting complex is contractible. This is shown in Ronan, "Lectures on buildings."

Hecke algebras:

7. (*) Let (W, S) be a dihedral Coxeter group. That is

$$W = \langle s, t \mid s^2 = t^2 = (st)^{m_{st}} = e \rangle$$

where $e \in W$ is the identity, and $m_{st} \in \{2, 3, 4, \ldots, \infty\}$. Given $0 \le m \le m_{st}$ write $st(m)$ for the product $stst\ldots$ where m terms appear and similarly for $ts(m)$. For example $st(0) = e$, $ts(1) = t$, $st(2) = st$, $ts(3) = tst$ etc.

a) Give explict descriptions of all elements of W, and hence describe the Bruhat order on W explicitly.

b) For $1 \le m < m_{st}$ find an explicit formula for the products

$$\underline{H}_s \underline{H}_{st(m)},\ \underline{H}_s \underline{H}_{ts(m)},\ \underline{H}_t \underline{H}_{ts(m)} \qquad \text{and} \qquad \underline{H}_t \underline{H}_{st(m)}$$

in terms of the Kazhdan-Lusztig basis. (*Hint:* Calculate the first few cases and then use induction. Use caution with small m.)

c) Conclude that $h_{x,y} = v^{\ell(y)-\ell(x)}$ for all $x \le y \in W$.

d) When m_{st} is finite, one has $\underline{H}_{st(m_{st})} = \underline{H}_{ts(m_{st})}$, which gives an algebraic relation between $\underline{H}_s$ and $\underline{H}_t$. This is the analog of the braid relation for the Kazhdan-Lusztig presentation of $\mathbf{H}$. Write down this equation for $m_{st} \le 6$. Can you find a reasonable formula for the coefficients?

8. (*) Let $W = S_4$, the symmetric group on $\{1,2,3,4\}$. Then W has the structure of a Coxeter group with $S = \{s_1, s_2, s_3\}$ where s_i denotes the transposition $(i, i+1)$.

a) Compute reduced expressions for all elements of W.

b) Calculate the Kazhdan-Lusztig basis $\{\underline{H}_x \mid x \in W\}$. How many non-trivial Kazhdan-Lusztig polynomials are there?

9. (*) Let W be a Weyl group of type D_4 with generating reflections s, t, u, v such that s, u, v all commute. Let $\underline{w} = suvtsuv$.

a) Use the defect formula to write the element $H_{\underline{w}}$ in terms of the standard basis.

b) Write the element $\underline{H}_{\underline{w}}$ in terms of the Kazhdan-Lusztig basis.

c) Hence compute the Kazhdan-Lusztig polynomial $h_{suv,suvtsuv}$.

10. (*) Some miscellaneous exercises from lecture.

a) Compute H_s^{-1}, and show that $\underline{H}_s$ is self-dual. Confirm that $\underline{H}_s^2 = \underline{H}_s(v + v^{-1})$.

b) Compute H_{st}^{-1} in terms of the standard basis. Given $w \in W$, for which $y \in W$ can there be a non-zero coefficient of H_y in the expression for H_w^{-1}? In the expression for $\overline{H_w}$? In the expression for $\omega(H_w)$?

c) Prove the uniqueness of the KL basis.

d) Find a formula for $H_w \underline{H}_s$.

e) Extrapolate the construction from lecture into a proof of the existence of the KL basis.

f) Prove the Deodhar formula.

11. Let (W, S, L) be a Coxeter system with a weight function. The *Hecke algebra with unequal parameters* $\mathbf{H}(W, S, L)$ is the $\mathbb{Z}[v^{\pm 1}]$-algebra generated by H_s, $s \in S$, subject to the usual braid relation and a new quadratic relation:

$$(H_s + v^{L(s)})(H_s - v^{-L(s)}) = 0.$$

Most features of the standard Hecke algebra extend to the Hecke algebra with unequal parameters, especially when the weight function is *positive* (i.e. $L(s) > 0$ for all $s \in S$).

a) Compute H_s^{-1}. Find the correct definition for a self-dual element $\underline{H}_s$.

b) Find a formula for $H_w \underline{H}_s$.

c) When L is positive, prove the existence and the uniqueness of the KL basis. What extra complication is required in the inductive construction?

d) Modify the Deodhar formula.

12. Continuing Q11: Let $\{s, t\}$ be the generators of a Coxeter group of type B, with $m_{st} = 4$, and let $L(s) = 1$ and $L(t) = 2$. Let $\mathbf{H}$ denote the Hecke algebra with unequal parameters. Compute the KL basis. Note that some KL polynomials have negative coefficients! This does not happen for usual Hecke algebras, as we will prove in this workshop.

13. Continuing Q11: When one Coxeter group (W, S, L) embeds inside another (W', S', ℓ) as the invariants under a diagram automorphism, one might expect there to be a corresponding relationship between their Hecke algebras (with unequal parameters). However, the relationship is quite subtle. We quickly explore this when (W', S') is $A_1^{\times n}$, and σ is the diagram automorphism which permutes the copies of A_1 cyclically. Therefore, (W, S, L) has type A_1, and $L(s) = n$.

a) Compute the self-dual generator of $\mathbf{H}(W', S', \ell)^\sigma$, and its square.

b) Compute the self-dual generator of $\mathbf{H}(W, S, L)$, and its square.

c) When $n = p$ is prime, show that these algebras are isomorphic modulo p.

14. Some more questions from lecture, dealing with the standard trace and standard pairing.

a) Compute $\varepsilon(H_x H_y)$. When is it non-zero?

b) Show that $\varepsilon(ab) = \varepsilon(ba)$.

c) Show that the standard basis is orthonormal for the standard pairing.

d) Show that the KL basis is graded orthonormal for the standard pairing.

15. (*) Let $\underline{w} = s_1 \dots s_m$ denote an expression. We write $x \le \underline{w}$ if there exists a subexpression $\mathbf{e}$ of $\underline{w}$ with $x = \underline{w}^{\mathbf{e}}$ (for example $\{x \in W \mid x \le \underline{w}\} = \{x \in W \mid x \le w\}$ if $\underline{w}$ is reduced). Given two subexpressions $\mathbf{e}, \mathbf{e}'$ of $\underline{w}$ let $x_0, x_1, \dots$ and $x'_0, x'_1, \dots$ be their Bruhat strolls (e.g. $x_i := s_1^{e_1} \dots s_i^{e_i}$). We define the *path dominance order* on subexpressions by saying that $\mathbf{e} \le \mathbf{e}'$ if $x_i \le x'_i$ for $1 \le i \le \ell(\underline{w})$. Show that for any $x \le \underline{w}$ there is a unique subexpression $\mathbf{e}$ of $\underline{w}$, the *canonical subexpression*, which is uniquely characterised by the following equivalent conditions:

a) $\mathbf{e} \le \mathbf{e}'$ for any subexpression $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$ (i.e. $\mathbf{e}$ is the unique minimal element in the path dominance order).

b) $\mathbf{e}$ has no D's in its UD labelling.

c) $\mathbf{e}$ is of maximal defect amongst all subexpressions $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$.

(If you know about Bott-Samelson resolutions: What geometric fact does the existence of $\mathbf{e}$ correspond to?)