Symmetric Functions: Problem Set 7

1. Let $\{u_\lambda\}, \{u'_\lambda\}$ (and similarly $\{v_\lambda\}, \{v'_\lambda\}$) be a pair of dual bases of Λ with respect to the form $\langle , \rangle$. If $\vec{u} = \vec{v}A$, prove that $\vec{v'} = \vec{u'}A'$.

2. Let $\lambda, \mu \in \mathrm{Par}(d)$. Prove that if λ is a refinement of μ, then λ occurs after μ in reverse lexicographic order. Is it true that μ dominates λ ?

3. Prove Pieri's rule (in the ring of symmetric functions in n variables, for large n): $s_\lambda\, h_k = \sum_{\mu \in \lambda \otimes k} s_\mu$ where $\lambda \otimes k$ is the set of partitions obtained by adding k boxes to the diagram of λ, no two in the same column. Deduce that the same equation holds in the ring Λ.

4. Let $\mathcal{A}$ denote the space of alternating polynomials in n variables. Show that (i) The a_γ for $\gamma \in D(n)$ span $\mathcal{A}$, (ii) multiplication by a_δ defines an isomorphism of vector spaces $\Lambda_n \to \mathcal{A}$. Thus, $\mathcal{A}$ is a free Λ_n-module generated by a_δ.

5. Let $\delta = (n-1, n-2, \cdots, 0)$. Show that $s_\delta = \prod_{1 \le i < j \le n}(x_i + x_j)$ in the ring Λ_n. Compute $s_{k\delta}$ for all $k \ge 1$.

6. Prove the identities:

$$f^\lambda = \sum_{\lambda \in \mu \otimes 1} f^\mu \tag{1}$$

$$(1 + |\lambda|) f^\lambda = \sum_{\mu \in \lambda \otimes 1} f^\mu \tag{2}$$

where f^λ is the number of standard Young tableaux of shape λ.