Symmetric Functions: Problem Set 6

1. Let $b \in \{m, h, p, s\}$. Let $n \geq 1$. Prove that $\{b_\lambda : \ell(\lambda) \leq n\}$ is a basis of Λ_n.

2. Let $n \geq 1$. Prove that $\{e_{\lambda'} : \ell(\lambda) \leq n\}$ is a basis of Λ_n.

3. What is the corresponding statement for the $\{f_\lambda\}$ (the forgotten symmetric functions)?

4. Prove that ω is an isometry (Hint: compute using s).

5. Using the symmetry property of the RSK algorithm, prove that

$$\prod_i (1 - tx_i)^{-1} \prod_{i<j} (1 - t^2 x_i x_j)^{-1} = \sum_{\lambda \in \mathrm{Par}} t^{|\lambda|} s_\lambda(x)$$

6. Prove that the matrix of ω in the m-basis is triangular. Do the same for the bases e, h, f.

7. Prove that

$$p_1^d = h_1^d = \sum_{\lambda \in \mathrm{Par}(d)} f^\lambda s_\lambda$$

   where f^λ is the number of standard Young tableaux of shape λ.

8. Use the above to show that f^λ is the dimension of $V(\lambda)$.