Symmetric Functions: Problem Set 5

1. Compute the action of the involution ω on each of the following: $m_\lambda, e_\lambda, h_\lambda, f_\lambda, p_\lambda, s_\lambda$.

2. Show by comparing the actions of ω on p_λ and s_λ that the number of even partitions of d (those for which the permutation of that cycle type is even) minus the number of odd partitions of d is equal to the number of self conjugate partitions of d.

3. Show that if $\lambda = 1^{m_1}2^{m_2}\cdots$ is a partition of d, the number of elements in S_d of cycle type λ is $d!/z_\lambda$ where $z_\lambda = 1^{m_1}m_1!2^{m_2}m_2!\cdots$.

4. Prove that

$$\prod_{i,j}(1-tx_iy_j)^{-1} = \sum_{\lambda\in\mathrm{Par}} t^{|\lambda|}s_\lambda(x)s_\lambda(y) \tag{1}$$

$$= \sum_{\lambda\in\mathrm{Par}} t^{|\lambda|}z_\lambda^{-1}p_\lambda(x)p_\lambda(y) \tag{2}$$

5. Prove that

$$\prod_{i,j}(1+tx_iy_j) = \sum_{\lambda\in\mathrm{Par}} t^{|\lambda|}s_\lambda(x)s_{\lambda'}(y) \tag{3}$$

$$= \sum_{\lambda\in\mathrm{Par}} t^{|\lambda|}z_\lambda^{-1}\epsilon_\lambda p_\lambda(x)p_\lambda(y) \tag{4}$$

where ϵ_λ is the sign of the permutation w_λ of cycle type λ.

6. Let $\{u_\lambda : \lambda \in \mathrm{Par}(d)\}$ and $\{v_\lambda : \lambda \in \mathrm{Par}(d)\}$ be bases of Λ^d for each $d \geq 1$. Let

$$\vec{u} = \vec{s}A \quad \text{and } \vec{v} = \vec{s}B$$

(there is one such A, B for each d). Suppose $AB' = I$ for each $d \geq 1$, show that:

$$\prod_{i,j}(1-tx_iy_j)^{-1} = \sum_{\lambda\in\mathrm{Par}} t^{|\lambda|}u_\lambda(x)v_\lambda(y)$$