Symmetric Functions: Problem Set 4

1. Let $p_0 = 0$ and $p_n = \sum_i x_i^n$ for $n \geq 1$. Define

$$P(t) = \sum_{n \geq 0} p_n \, t^n$$

Show the following:

$$P(t) = \sum_i \frac{tx_i}{1 - tx_i} = t \frac{d}{dt} \sum_i \log(1 - tx_i)^{-1} \quad (1)$$

$$P(t) = t \frac{H'(t)}{H(t)} \quad (2)$$

$$P(-t) = -t \frac{E'(t)}{E(t)} \quad (3)$$

$$\sum_{r=1}^{n} p_r \, h_{n-r} = n h_n \quad (n \geq 1) \quad (4)$$

$$\sum_{r=1}^{n} (-1)^{r-1} p_r \, e_{n-r} = n h_n \quad (n \geq 1) \quad (5)$$

2. Prove that $\omega(p_r) = (-1)^{r-1} p_r$ and that $\omega(p_\lambda) = \epsilon_\lambda p_\lambda$ where ϵ_λ is the sign of w_λ, an element of the symmetric group of cycle type λ.

3. The trace of the involution ω on Λ^d is the number of even partitions of d minus the number of odd partitions of d (a partition λ is even if $\epsilon_\lambda = 1$ and odd otherwise).

4. Let the *graded trace of* ω *be:*

$$\zeta(q) = \sum_{d \geq 0} Tr \, (\omega|_{\Lambda^d}) \, q^d$$

Prove that $\zeta(q) = \dfrac{1}{(1-q)(1+q^2)(1-q^3)(1+q^4)\cdots}$.

5. Using the fact that

$$\frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} = (1+q)(1+q^2)(1+q^3)\cdots$$

prove that $Tr \, (\omega|_{\Lambda^d})$ equals the number of self conjugate partitions of d.