Symmetric Functions: Problem Set 3[1]

1. Prove that the map $\omega : e_i \mapsto h_i$ defined in lecture is indeed an involution (we only did this by example).

2. Define $f_\lambda := \omega(m_\lambda)$. The f_λ are called the *forgotten symmetric functions.* Show that they form a basis of Λ. Compute f_λ for $|\lambda| = 1, 2, 3$.

3. Compute the change of basis matrix when e_λ is written in terms of the f_μ.

4. Let $x_i = 1$ for $1 \leq i \leq n$ and $x_i = 0$ for $i > n$. Show that under this substitution, $e_r = \binom{n}{r}$. Find the values of h_r and m_λ for $r \geq 1$ and $\lambda \in \mathrm{Par}$.

5. Let us substitute $x_i = 1/n$ for $1 \leq i \leq n$ and $x_i = 0$ otherwise. Now take the limit as $n \to \infty$. Compute the values of e_r, h_r and m_λ in this limit.

6. Prove that:

$$e_n = \det\left(h_{1-i+j}\right)_{1\leq i,j\leq n}$$
$$h_n = \det\left(e_{1-i+j}\right)_{1\leq i,j\leq n}$$

7. If $h_n = n$ for each $n \geq 1$ (recall that the h_n are algebraically independent generators, and one can assign any value to them), the sequence $(e_n)_{n\geq 1}$, is periodic with period 3.

[1]Reference: I.G. Macdonald, *Symmetric Functions and Hall Polynomials.*