Symmetric Functions: Problem Set 2

1. Expand the following product and write it as a power series in t with coefficients being symmetric functions in $\underline{x}$:

$$\prod_{i=1}^{\infty}(1+tx_i)$$

2. Expand the following product, writing it as a power series in t, with coefficients being products of symmetric functions of $\underline{x}$ and $\underline{y}$:

$$\prod_{i,j=1}^{\infty}(1+tx_iy_j)$$

3. Let $e_n(\underline{x})$ denote the monomial symmetric function $m_{(1,1,\cdots,1)}(\underline{x})$. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ of d, define:

$$e_\lambda(\underline{x}) = \prod_{i=1}^{p} e_{\lambda_i}(\underline{x}).$$

Write $e_\lambda(\underline{x})$ as a linear combination of the basis $\{m_\mu(\underline{x}) : \mu \in \mathrm{Par}(d)\}$ for $d = 1, 2, 3, 4$ and each choice of λ.

4. Expand the following product and write it as a power series in t with coefficients being symmetric functions in $\underline{x}$:

$$\prod_{i=1}^{\infty}(1-tx_i)^{-1}$$

5. Expand the following product, writing it as a power series in t, with coefficients being products of symmetric functions of $\underline{x}$ and $\underline{y}$:

$$\prod_{i,j=1}^{\infty}(1-tx_iy_j)^{-1}$$

6. Let $h_n(\underline{x}) := \sum_{\mu\in\mathrm{Par}(n)} m_\mu(x)$ denote the *complete homogeneous symmetric function.* For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ of d, define:

$$h_\lambda(\underline{x}) = \prod_{i=1}^{p} h_{\lambda_i}(\underline{x}).$$

Write $h_\lambda(\underline{x})$ as a linear combination of the basis $\{m_\mu(\underline{x}) : \mu \in \mathrm{Par}(d)\}$ for $d = 1, 2, 3, 4$ and each choice of λ.

7. Let $n \geq 1$; recall that we have a map $\rho : \Lambda^d \to \Lambda_n^d$ obtained by setting all variables $x_i, i > n$ equal to zero. If $\{b_i\}$ is a basis of Λ^d, show that it is not true that the nonzero elements in $\{\rho(b_i)\}$ must be a basis of Λ_n^d ? Can you give an example of a basis of Λ^d for which this does hold ?

8. Express the product $m_{(2,1)}\, m_{(1,1)}$ as a linear combination of monomial symmetric functions.

9. For positive integers p, q show that

$$m_{(1^p)} m_{(1^q)} = \sum_{k=0}^{\min(p,q)} \binom{p+q-2k}{p-k} m_{(2^k, 1^{p+q-2k})}$$

10. Prove that m_λ can be written as a polynomial in the $e_1, e_2, \cdots$, i.e, there is a polynomial $f_\lambda \in k[u_1, u_2, \cdots]$ such that $m_\lambda = f(e_1, e_2, \cdots)$.