Symmetric Functions: Problem Set 11

1. Complete the proof of the hook-content formula by simplifying the product obtained in class into the required form.

2. Prove that the skew Schur functions are symmetric functions by adapting the argument used for Schur functions (*free and paired occurrences of* i, $i+1$ *etc*).

3. Prove that:
$$\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$$

4. Let μ, ν be partitions of d; thus $V(\mu)$, $V(\nu)$ and $V(\mu) \otimes V(\nu)$ are all representations of S_d. Define:
$$\chi_\mu * \chi_\nu := \chi_{V(\mu)\otimes V(\nu)}$$
We have
$$\chi_\mu * \chi_\nu = \sum_{\lambda \in \mathrm{Par}(d)} g_{\mu\nu}^{\lambda}\, \chi_\lambda$$
The $g_{\mu\nu}^{\lambda}$ are nonnegative integers, called the *Kronecker coefficients*. Prove that $g_{\mu\nu}^{\lambda}$ is symmetric in μ, ν, λ.

5. Let x, y respectively denote the infinite list of variables $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$. Prove that:
$$s_\lambda(xy) = \sum_{\mu,\nu} g_{\mu\nu}^{\lambda}\, s_\mu(x) s_\nu(y)$$
where xy denotes the list of variables $x_i y_j$ for $(i,j) \in \mathbb{N} \times \mathbb{N}$. *Hint: Expand in terms of power sums first.*

6. Prove :
$$\prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda,\mu,\nu} g_{\mu\nu}^{\lambda}\, s_\lambda(x) s_\mu(y) s_\nu(z)$$

7. Define
$$\zeta := \sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\mu h_{mu}(x) m_\mu(y)$$
each side being equal to $\prod_{i,j} \dfrac{1}{1 - x_i y_j}$.

(a) Verify that $\zeta = \sum_\alpha y^\alpha h_\alpha(x)$ where the sum runs over all possible multi-indices α.

(b) This identity holds if we restrict x and y to finite lists of variables $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$. The sums on both sides then run over partitions with at most n parts (since $s_\lambda = 0 = m_\mu$ if λ, μ have more than n parts). Show that $s_\lambda(x)$ is the coefficient of $y^{\lambda+\delta}$ in $\zeta a_\delta(y)$.

(c) Recall $a_\delta = \sum_{w \in S_n} \mathrm{sgn} w \; y^{w\delta}$. Use this to show that:

$$s_\lambda(x) = \sum_{w \in S_n} \mathrm{sgn} w \; h_{\lambda+\delta-w\delta} \tag{1}$$

where h_i is taken to be zero for $i < 0$.

(d) Prove that the above expression is nothing but the determinant:

$$\det \left(h_{\lambda_i+j-i}\right)_{i,j=1}^n$$

This identity is called the *Jacobi-Trudi* identity.

(e) Derive an analogous expression for s_λ in terms of the e_k.

(f) Use equation (1) to give an expression for the entries $K^{-1}_{\lambda\mu}$ of the inverse Kostka matrix.