Symmetric Functions: Problem Set 1[1]

1. Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a partition (written as an infinite list, with all but finitely many entries zero). Let $m \geq \lambda_1, n \geq \lambda_1'$. Show that the $m+n$ numbers

$$\lambda_i + n - i \quad (1 \leq i \leq n), \qquad n - 1 + j - \lambda_j' \quad (1 \leq j \leq m)$$

   are a permutation of $\{0, 1, 2, \cdots, m+n-1\}$.

2. Let λ, μ be partitions of n such that λ *covers* μ in the dominance order, i.e., $\lambda > \mu$ and if ν is such that $\lambda \geq \nu \geq \mu$, then $\nu = \lambda$ or $\nu = \mu$. Show that λ can be obtained by removing one box from the j^{th} row of μ and moving it to the i^{th} row, for some $i < j$.

3. A matrix of non-negative real numbers is said to be doubly stochastic if its row and column sums are all equal to 1. Let λ, μ be partitions of n. Show that λ dominates μ if and only if there exists a doubly stochastic $n \times n$ matrix M such that $M\lambda = \mu$ (where λ, μ are regarded as column vectors of length n).

4. Let λ be a partition. The *hook-length* of λ at $x = (i,j) \in \lambda$ is defined to be

$$h(x) = (\lambda_i - i) + (\lambda_j' - j) + 1.$$

   The *content* of x is defined to be $c(x) = j - i$. Prove that

$$\sum_{x \in \lambda} (h(x)^2 - c(x)^2) = |\lambda|^2.$$

5. Show that the set of partitions of n under the dominance order is a *lattice*. In other words, each pair of partitions of n has a greatest lower bound and a least upper bound.

6. Let $m \geq 1$.

   (a) Show that the set $\mathcal{D}(m)$ of strictly decreasing m-tuples of non-negative integers is in bijection with the set of partitions with at most m parts under the map $\lambda \mapsto \lambda^\dagger$ where $\lambda_i^\dagger = \lambda_i - (m-i)$ for all i.

[1] Reference: I.G. Macdonald, *Symmetric Functions and Hall Polynomials.*

(b) Let $\lambda \in \mathcal{D}(m)$, $1 \leq i \leq m$ and $p \geq 1$. Define $u_j = \lambda_j$ for $j \neq i$, and $u_i = \lambda_i - p$. Assume that the $\{u_j : 1 \leq j \leq m\}$ are all distinct, non-negative integers. Let $\mu \in \mathcal{D}(m)$ denote the tuple obtained by rearranging the u_j in descending order. Describe the image $\mu^\dagger$ of μ under the above bijection.

7. Let $\mathcal{P}(n)$ denote the set of partitions of n, and $\mathbb{N}$ the set of positive integers. For each $r \geq 1$, let

$$a(r,n) = \#\{(\lambda, i) \in \mathcal{P}(n) \times \mathbb{N} : \lambda_i = r\}$$
$$b(r,n) = \#\{(\lambda, i) \in \mathcal{P}(n) \times \mathbb{N} : m_i(\lambda) \geq r\}$$

Show that

$$a(r,n) = b(r,n) = p(n-r) + p(n-2r) + \ldots$$

where $p(m)$ is the number of partitions of m. Deduce that

$$\prod_{\lambda \in \mathcal{P}(n)} \prod_{i \geq 1} i^{m_i(\lambda)} = \prod_{\lambda \in \mathcal{P}(n)} \prod_{i \geq 1} m_i(\lambda)!$$

8. Keep the above notation. Let $h(r,n) = \#\{(\lambda, x) : \lambda \in \mathcal{P}(n), x \in \lambda \text{ and } h(x) = r\}$, where $h(x)$ is the hook-length of λ at x. Show that $h(r,n) = ra(r,n)$.