Exercise set 6

(1) (Difficulty level 3) Given a large supply of round beads of two differnt colours, how many distinct necklaces each with 8 beads can you make? How many with 9 beads? With 10 beads? ...

(2) (Difficulty level 2) If a finite group acts transitively on a set X with at least 2 elements, then there exists an element of G that does not fix any element.

(3) (Difficulty level 1) Given a conjugacy class with at least 2 elements of a finite group G, there always exists an element of G that does not commute with any element of the conjugacy class.

(4) (Difficulty level 1) The conjugates of a finite index proper subgroup cannot cover a group.

(5) (Difficulty level 3 after hint) Let G and H be finite groups. Show that the complex irreducible representations of $G \times H$ are precisely those of the form $V \otimes W$, where V and W are irreducible representations respectively of G and H. (Hint: Use Burnside's lemma, the one which says that the algebra homomorphism $\mathbb{C}G \to \mathrm{End}_{\mathbb{C}} V$ defining an irreducicble representation V is surjective.)

(6) (Difficulty level 2) Show by means of an example that the assertion of the previous item is not true if we replace $\mathbb{C}$ by $\mathbb{R}$.