Exercise set 4

The symbol G will denote a finite group. The character of a finite dimensional (complex) representation V is denoted by $\mathscr{T}_V$.

(1) (Difficulty level: 1) Convince yourself of the following:
- $\mathrm{Hom}(V,W) \simeq V^* \otimes W$
- $\mathscr{T}_{V\oplus W} = \mathscr{T}_V + \mathscr{T}_W$
- $\mathscr{T}_{V\otimes W} = \mathscr{T}_V \mathscr{T}_W$
- $\mathscr{T}_{V^*} = \overline{\mathscr{T}_V}$
- $\mathscr{T}_{\mathrm{Hom}(V,W)} = \overline{\mathscr{T}_V}\mathscr{T}_W$

(2) (Difficulty level: 3) Calculate character tables of the symmetric groups on 3, 4, and 5 letters; of cyclic groups; of the dihedral groups; of the non-cyclic group of order 4; of all groups of order 8; of all groups up to order 12; etc.

(3) (Difficulty level: 2) Deduce the following from Wedderburn's structure theorem, where R denotes a finite dimensional semisimple algebra over an algebraically closed field k:
- (a criterion for simplicity) R is simple if and only if it admits a unique simple module.
- (density) If $V_1, \ldots, V_k$ are pairwise non-isomorphic simple R-modules, and $\varphi_1, \ldots, \varphi_k$ are arbitrarily specified k-linear transformations, then there exists r in R which acts on V_i like φ_i for all $1 \leq i \leq k$.
- ("left semisimplicity and right semisimplicity are equivalent") R is semisimple as a right module.

Additional problems (Optional)

(1) (Burnside's lemma) Let k be an algebraically closed field. Let R be a k-algebra (not necessarily finite dimensional) and let V be a finite dimensional simple R-module. Show that $R \to \mathrm{End}_k V$ (the map defining V as an R-module) is surjective. (Hint: Let the image of R in $\mathrm{End}_k V$ be denoted by S. Note that V is a simple module for S. Since $\mathrm{End}_k V = V \oplus \cdots \oplus V$ ($\dim_k V$ times) as a $\mathrm{End}_k V$ module, it follows that it is semisimple as an S-module. Thus S is semisimple as a module over itself (being a submodule of a semisimple module), which means that S is a semisimple algebra. Since V is a simple module for S, it follows from Wedderburn that $\dim_k S \geq (\dim_k V)^2$. Thus $S = \mathrm{End}_k V$. $\square$)

(2) Let k be an algebraically closed field. Show that any finite dimensional simple k-algebra is isomorphic to the matrix ring $M_n(k)$ for some n. (Hint: Let R be such an algebra. By definition, $R \neq 0$. Choose a simple module V for R and consider $R \to \mathrm{End}_k V$ defining V as an R-module (note that V exists). This map is surjective by Burnside's lemma (previous item) and injective because R is simple. $\square$)