Exercise set 3

Throughout k denotes a field and A a k-algebra.

(1) (Difficulty level: 2) Prove or disprove: a k-vector space V is a simple module for $\mathrm{End}_k(V)$.

(2) (Difficulty level: 2) Every simple module for a finite dimensional k-algebra is finite dimensional. (In particular, every irreducible representation of a finite group is finite dimensional.)

(3) (Difficulty level: 2) Show that the group ring kG is isomorphic to its opposite (for any field k and any group G).

(4) (Difficulty level: 3) Show that every simple linear representation of a finite abelian group over an algebraically closed field is one dimensional. (Hint: Use Schur's lemma.)

(5) (Difficulty level: 2) Show by means of an example that the hypothesis in item (4) above of the field being algebraically closed cannot be omitted.

(6) (Difficulty level: 2) Let k be algebraically closed and let V and W be finite dimensional semisimple A-modules. Show that if $\dim \mathrm{End}_A V = \dim \mathrm{Hom}_A(V, W) = \dim \mathrm{End}_A(W)$, then V is isomorphic to W.

(7) (Difficulty level: 2) Let k be algebraically closed and M be a semisimple A-module. Show the following:
   - M is simple if and only if $\mathrm{End}_A(M) \simeq k$
   - M is multiplicity free (that is, no simple component of M occurs with multiplicity more than 1) if and only if $\mathrm{End}_A(M)$ is commutative.

(8) (Difficulty level: 3) Let M be a multiplicity free semisimple module (over some ring). Determine the submodules of M.

(9) (Difficulty level: 2) (Converse of Maschke) Let k be a field, G a group, and kG the group ring. We can turn k into an kG-module by letting each g in G act as identity and extending linearly: $(\sum \lambda_g g) \cdot \mu = (\sum \lambda_g)\mu$. This module is called the *trivial* kG-module. Let $H := \{\sum_{g \in G} \lambda_g g \in kG \mid \sum_{g \in G} \lambda_g = 0\}$. Then H is a codimension 1 subspace of kG and is an kG-submdoule. Moreover, kG/H is trivial.

   Now assume that G is finite. Show that the span of $\sum_{g \in G} g$ is the only 1-dimensional trivial kG-submodule of kG. Conclude that H does not have a complemtary submodule if the characteristic of k is positive and divides $\#G$.

(10) (Difficulty level: 3) Let F be the finite field $\mathbb{Z}/p\mathbb{Z}$ and $M_3(F)$ the ring of 3×3 matrices with coefficients in F. What are the possible dimensions of left ideals in $M_3(F)$? How many left ideals are there of each such dimension?