Exercise set 2.6

(1) (Difficulty level 2) Show that the matrix of the character table of a finite group (over the complex numbers) is invertible. More precisely, show that the absolute value of the determinant of the character table is $\prod_g \sqrt{z_g}$, where the product is over a set of representatives g of the conjugacy classes and z_g is the cardinality of the centralizer of g.

(2) (Difficulty level 2) Show that the restriction to an invariant subspace of a diagonalizable linear operator on a finite dimensional vector space is diagonalizable. Suppose that a representation of a group breaks up into a direct sum of 1-dimensional representations. Show that any invariant subspace also breaks up into a direct sum of 1-dimensional representations.

(3) (Difficulty level 3) Given a weak composition λ of n with m parts, we have naturally associated to it a 1-dimensional representation homogeneous of degre n of the torus $T_m(K)$ by mapping $\Delta(x_1, \ldots, x_n)$ to the monomial corresponding to λ. Show that this is a bijective correspondence (between such weak compositions and such 1-dimensional represention).

(4) (Difficulty level 2) Let m be a positive integer and let λ be a partition with at most m parts. Consider the irreducible polynomial representation W_λ of $GL_m(\mathbb{C})$. Its character is $s_\lambda(x_1, \ldots, x_m)$, where s_λ is the Schur function corresponding to λ. Given a weak composition μ of n with m parts, let mon_μ be the corresponding monomial of degree n in the variables $x_1, \ldots, x_m$. We call μ a *weight* of λ if the coefficient of mon_μ is $s_\lambda(x_1, \ldots, x_m)$ is not zero. The coefficient itself (in case it is not zero) is called the *multiplicity* of the weight μ in W_λ.
- Let λ be a partition of n with at most 2 parts. Determine the weights and their multiplicities in the $GL_2(\mathbb{C})$ module W_λ.
- Determine the weights and their multiplicities of the $GL_m(\mathbb{C})$-modules $W_{(n)}$ and $W_{(1^n)}$.

(5) (Difficulty level 3) Let $V = (K^m)^{\otimes n}$, where K is a field. Let ρ be the representation of $\mathfrak{S}_n$ on V, and θ be the representation of $GL_m(K)$ on V. Let w be in $\mathfrak{S}_n$. Show that the trace of $\rho(w) \cdot \theta(\Delta(x_1, \ldots, x_m))$ equals $\sum_\lambda \chi_\lambda(w) s_\lambda(x_1, \ldots, x_m)$, as the sum varies over all partitions λ of n.

Hint: This is Lemma 6.5.1 in Amri's book. We have $V \simeq K[I(n,m)]$ (as $\mathfrak{S}_n$-modules) where $I(n,m) = I$ denotes the set of all functions from $[n]$ to $[m]$, for the standard basis of $(K^m)^{\otimes n}$ is indexed by I. Note that the standard basis elements form common eigenvectors for the action of the torus (diagonal subgroup of GL_m), and are permuted by elements of $\mathfrak{S}_n$. Let W be the set of weak compositions of n with m parts. There is a "type map" $I \to W$, and two elements of I belong to the same orbit of $\mathfrak{S}_n$ if and only if their types are the same. So we may write $K[I] = \oplus_\tau K[I_\tau]$, where τ varies over W and I_τ is the fibre over τ of the type map. On each $K[I_\tau]$, each element of the torus acts like a scalar, like the monomial corresponding to τ.

From W to the set P of partitions of n we have a natural map: put the constituents of the weak composition in weakly decreasing order. For a partition μ of n, the $\mathfrak{S}_n$-representation $K[X_\tau]$ is isomorphic to $K[X_\mu]$ for all τ in the fibre in W over μ. Let us fix μ in P and consider $\sum_{\tau\in W, \tau\mapsto\mu} K[I_\tau]$. The trace of $\rho(w) \cdot \theta(\Delta(x_1, \ldots, x_m))$ on this is $\mathrm{Trace}(w, K[X_\mu]) \cdot m_\mu(x_1, \ldots, x_m)$.

Now $K[X_\mu] = \sum_\nu V_\nu^{\oplus K_{\nu\mu}}$. So the required trace is

$$\sum_\mu \sum_\nu K_{\nu\mu}\chi_\nu(w) m_\mu(x_1, \ldots, x_m)$$

$$= \sum_\nu \chi_\nu(w) \left(\sum_\mu K_{\nu\mu} m_\mu(x_1, \ldots, x_m) \right)$$

$$= \sum_\nu \chi_\nu(w) \cdot s_\nu(x_1, \ldots, x_m)$$