Exercise set 2.5

Throughout, K denotes a field and $I(n,m)$ the set of maps from $[n]$ to $[m]$. There is a natural action of $\mathfrak{S}_n$ on $I(n,m)$.

(1) (Difficulty level 2) Consider a vector space over K of dimension m, say K^m. Consider $(K^m)^{\otimes n} = K^m \otimes \cdots \otimes K^m$ (n times). Let $\mathfrak{S}_n$ act on this by permuting the factors $\sigma(v_1 \otimes \cdots v_m) := v_{\sigma^{-1}1} \otimes \cdots \otimes v_{\sigma^{-1}m}$. Show that $K[I(n,m)] \simeq (K^m)^{\otimes n}$. Conclude that:

$$S_K(m,n) \simeq \mathrm{End}_{\mathfrak{S}_n}((K^m)^{\otimes n})$$

(2) (Difficulty level 3) Let R be the K-algebra $M_2(K) \times M_3(K)$. Let φ be the inclusion of R in $M_{11}(K)$ given by

$$(A,B) \mapsto \text{block diagonal } (A,A,A,A,B)$$

Let C denote the commutant in $M_{11}(K)$ of the image $\varphi(R)$. Show that C is a semisimple algebra. Determine its Wedderburn decomposition. How does the defining representation K^{11} of $M_{11}(K)$ decompose as a C-module? Determine C as a subset of $M_{11}(K)$.

(3) (Difficulty level 2) Recall that an element of $(K^m)^{\otimes n}$ is said to be a *symmetric n-tensor* if it is invariant under $\rho(w)$ for every w in $\mathfrak{S}_n$. Assuming K to be algebraically closed of characteristic 0, show that the space of symmetric tensors is a simple $GL_m(K)$-module of dimension equal to the number of weak compositions of n with at most m parts.

(4) (Difficulty level 2) Recall that an element of $(K^m)^{\otimes n}$ is said to be an *alternating n-tensor* if it is transforms under $\rho(w)$ by the sign of w, for every w in $\mathfrak{S}_n$. Assuming K to be algebraically closed of characteristic 0, show that the space of alternating tensors is either zero (iff $m < n$) or is a simple $GL_m(K)$-module of dimension equal to $\binom{m}{n}$.

(5) (Difficulty level 3) Let K be algebraically closed of characteristic 0. Determine the partitions λ with at most m parts for which the corresponding irreducible $GL_m(K)$-module W_λ is one dimensional. Show that the non-negative integral powers of the determinant are the only 1-dimensional polynomial representations of $GL_m(K)$.

(6) (Difficulty level 2) Compute the dimensions and characters of the representation $W_{2,1}$ of $GL_2(\mathbb{C})$ and the representation $W_{2,1}$ of $GL_3(\mathbb{C})$.

(7) (Difficulty level 3) Show that the set of diagonalizable $m\times m$ complex matrices is dense (with respect to the usual topology) in the space of all $m \times m$ complex matrices. Deduce that the set of diagonalizable invertible $m \times m$ matrices is dense in the space of all invertible $m \times m$ complex matrices. State and prove an analogous statement over an arbitrary algebraically closed field with respect to the "Zariski topology".

(8) (Difficulty level 3) Recall that the "convolution" product of two elements α and β of $S_K(m)$ is defined as follows:

$$(\alpha \star \beta)(f) = \iint f(xy)\, d\alpha(x)\, d\beta(y)$$

Show that the order of integration in the above can be reversed, or in other words that:

$$\iint f(xy)\, d\alpha(x)\, d\beta(y) = \iint f(xy)\, d\beta(y)\, d\alpha(x)$$

(9) (Difficulty level 2) Express the element δ_I (where I stands for the $m \times m$ identity matrix) as a linear combination of the "standard basis" $\epsilon_{\underline{i},\underline{j}}$ of $S_K(m)$ where $(\underline{i},\underline{j})$ varies over a set of representatives of the $\mathfrak{S}_n$-orbits of $I(n,m) \times I(n,m)$.