Exercise set 2.4

Throughout, K denotes a field. We denote:

- by $I(n,m)$ the set of maps from $[n]$ to $[m]$,
- by $OP(n,m)$ the set of partitions of $[n]$ into m ordered parts.

These sets come with natural actions of $\mathfrak{S}_n$ on them.

(1) (Difficulty level 1) Let X be a finite set on which a group G acts. Show that KX (the free K-vector space generated by X) and $K[X]$ (the space of K-valued functions on X) are isomorphic as G-representations via $x \mapsto \delta_x$.

(2) (Difficulty level 1) Show the following: for elements $\underline{p}$ and $\underline{q}$ of $I(n,m)$ and A, B matrices of size $m \times m$ (with entries over some field or commutative ring), $(AB)_{\underline{p},\underline{q}} = \sum_{\underline{t} \in I(n,m)} A_{\underline{p},\underline{t}} \cdot B_{\underline{t},\underline{q}}$.

(3) (Difficulty level 1) Describe a "natural" bijection between $I(n,m)$ and $OP(n,m)$ that preserves type and is $\mathfrak{S}_n$-equivariant.

(4) (Difficulty level 1) Let λ be a weak composition of n into m parts. Let λ^1 be the partition of n into at most m parts obtained by putting the constituents of λ in weakly decreasing order. Let X_λ be the set of all ordered partitions of n into m parts with type λ. Observe that X_λ and X_{λ^1} are isomorphic as $\mathfrak{S}_n$-sets.

(5) (Difficulty level 3) Work out the multiplicative structure constants of $S_K(m,n)$ for some small values of m and n.