Exercise set 2.2

Throughout, K denotes a field and V a vector space over K of finite dimension $d \geq 1$. We fix an integer $m \geq 1$ and denote by G the group $GL_K(m)$. We denote by $A_K(m)$ the ring of polynomial functions on G and by $S_K(m)$ and $S_K(m,n)$ the appropriate Schur algebras (as defined in the lecture).

(1) (Difficulty level 1) Let U and W be finite dimensional K-vector spaces. Convince yourself that the following are equivalent for a (set) map $\varphi : U \to W$:
   (a) There exists a basis $\{w_i\}$ of W such that the K-valued functions φ_i on U defined by $\varphi(u) = \sum_i \varphi_i(u) w_i$ are all polynomial.
   (b) For any basis $\{w_i\}$ of W, the K-valued functions φ_i on U defined by $\varphi(u) = \sum_i \varphi_i(u) w_i$ are all polynomial.
   (c) For any linear functional ζ on W, the K-valued function $\zeta \circ \varphi$ on U is polynomial.
   (d) For any polynomial function f on W, the K-valued function $f \circ \varphi$ on U is polynomial.

   If any of these holds, then φ is said to be a *polynomial map* from U to W.

(2) (Difficulty level 2) Let $\rho : G \to GL(V)$ be a polynomial representation and $\rho' : G \to GL(V^*)$ the contragredient representation of ρ. Observe that $g \mapsto \rho'(g)^{-1}$ is a polynomial map from G to $\operatorname{End}_K V^*$ (although not a group homomorphism, but only an anti-homomorphism).

(3) (Difficulty level 1) Verify that $S_K(m,1)$ is isomorphic to the matrix algebra $M_m(K)$.

(4) (Difficulty level 2) For α in $S_K(m)$, let α_n denote its projection to $S_K(m)$. (Recall that this projection is induced from the inclusion of homogeneous polynomials of degree n in the ring $A_K(m)$.) We think of $\alpha \mapsto \alpha_n$ as a map from $S_K(m)$ to $S_K(m)$, considering α_n to be an element of $S_K(m)$ under the inclusion of $S_K(m,n) \subseteq S_K(m)$. (Recall that this inclusion is induced by the projection of $A_K(m)$ onto its homogeneous component of degree n.) Show that $\delta_{I,n}$ (where I stands for the identity element of G, the $m \times m$ identity matrix), as n varies over the non-negative integers, are pairwise orthogonal central idempotents. (Caution: they are not primitive central idempotents, except in very special cases, as we will see.)

(5) (Difficulty level 3) We now outline a proof of the fact that a (finite dimensional) polynomial $S_K(m)$-module arises from a polynomial representation of G. More precisely, we show that any polynomial $S_K(m)$-module $\tilde{\rho} : S_K(m) \to \operatorname{End}_K(V)$ that is homogeneous of degree n arises from a homogeneous polynomial representation $\rho : G \to GL(V)$ of G of degree n. The candidate for ρ is clear: $\rho(x) := \tilde{\rho}(\delta_x)$ for x in G. It is also clear that ρ is a representation since $\delta_{xy} = \delta_x \star \delta_y$). It remains only to prove that $\langle \xi, \rho(g)v \rangle$ is a polynomial function of g, as g varies over G (for any fixed v in V and ξ in V^*).

   Since $S_K(m,n)$ is the dual of the finite dimensional K-vector space $A_K(m,n)$, it is clear that there exists a homogeneous polynomial $c_{\xi,v}$ in $A_K(m,n)$ such that

$$\langle \xi, \tilde{\rho}(\alpha_n)v \rangle = \int c_{\xi,v}(x)\, d\alpha_n(x) \quad \text{for all } \alpha_n \text{ in } S_K(m,n).$$

   Since $v = \tilde{\rho}(\delta_{I,n})v$, it follows that

$$\tilde{\rho}(\alpha)v = \tilde{\rho}(\alpha)(\tilde{\rho}(\delta_{I,n})v) = \tilde{\rho}(\alpha \star \delta_{I,n})v = \tilde{\rho}(\alpha_n)v \quad \text{for all } \alpha \text{ in } S_K(m).$$

   On the other hand, since $c_{\xi,v}$ is homogeneous of degree n, we have

$$\int c_{\xi,v}\, d\alpha(x) = \int c_{\xi,v}\, d\alpha_n(x) \quad \text{for all } \alpha \text{ in } S_K(m).$$

   From the equations in the last three displays, we conclude that

$$\langle \xi, \tilde{\rho}(\alpha)v \rangle = \int c_{\xi,v}(x)\, d\alpha(x) \quad \text{for all } \alpha \text{ in } S_K(m).$$

   Putting $\alpha = \delta_g$, we obtain $\langle \xi, \rho(g)v \rangle = c_{\xi,v}(g)$. □