Exercise set 2.2

Throughout, K denotes a field and V a vector space over K of finite dimension $d \geq 1$. We fix an integer $m \geq 1$ and denote by G the group $GL_K(m)$. We denote by $A_K(m)$ the ring of polynomial functions on G and by $S_K(m)$ and $S_K(m,n)$ the appropriate Schur algebras (as defined in the lecture).

(1) (Difficulty level 2) What is the dimension of the K-algebra $S_K(m,n)$?

(2) (Difficulty level 1) Show that $G \to S_K(m)$ given by $g \mapsto \delta_g$ is an injection. (Here and elsewhere, the "Dirac delta" δ_g denotes the linear functional on $A_K(m)$ given by $f \mapsto f(g)$.)

(3) (Difficulty level 2) Show that $\delta_g \star \delta_h = \delta_{gh}$ for g and h in G.

(4) (Difficulty level 3) Recall the following three definitions of the multiplication of two elements α and β in $S_K(m)$. Convince yourself that they are the same:
   (a) $(\alpha \star \beta)(f) = \iint f(xy)\, d\alpha(x)\, d\beta(y)$
   (b) $(\alpha \star \beta)(f) = \beta(y \mapsto \alpha(\rho_y f))$, where $\rho_y f(x) := f(xy)$.
   (c) $(\alpha \star \beta)(f) = \sum_i \alpha(f_i^1)\beta(f_i^2)$, where $\Delta(f) = \sum_i f_i^1 \otimes f_i^2$ is the image of f under the "coproduct" Δ, which is the K-algebra map from $A_K(m) \to A_K(m) \otimes A_K(m)$ given by $X_{ij} \mapsto \sum_k X_{ik} \otimes X_{kj}$

(5) (Difficulty level 2) Show that under the multiplication defined as in the previous item, the Schur algebra $S_K(m)$ becomes an associative unital K-algebra. What is its multiplicative identity?

(6) (Difficulty level 2) Recall that the inclusion of $S_K(m,n)$ in $S_K(m)$ is induced by the natural surjection of $A_K(m)$ onto its n-th degree component (which maps any polynomial to its homogeneous component of degree n). Show that $S_K(m,n)$ is a two-sided ideal of $S_K(m)$.

(7) (Difficulty level 2) By $\mathrm{Sym}(V)$ we mean the polynomial algebra generated in the variables $z_1, \ldots, z_d$, where $e_1, \ldots, e_d$ form a basis of V. Note that $\mathrm{Sym}(V)$ is a polynomial ring with the usual grading (where the variables $z_1, \ldots, z_d$ are all considered to be of degree 1). The graded piece of degree n is denoted by $\mathrm{Sym}^n(V)$. There is a natural action of $GL(V)$ on $\mathrm{Sym}(V)$ by algebra automorphisms that preserves degrees: for g in $GL(V)$, we let $gz_j = a_1(g)z_1 + \ldots + a_d(g)z_d$, where $ge_j = a_1(g)e_1 + \ldots + a_d(g)e_d$ (in other words, the action on the variables is the same as that on $e_1, \ldots, e_d$). The $\mathrm{Sym}^n(V)$ provide examples of polynomial representations of $GL(V)$.