Exercise set 2.1

Throughout K denotes a field and V a vector space over K of finite dimension $d \geq 1$.

(1) (Difficulty level 1) Convince yourself of the following facts:
   (a) The notion of a polynomial function on V is independent of the choice of a basis for V.
   (b) Any invariant subspace and any quotient of a polynomial representation of $GL(V)$ is a polynomial representation.
   (c) Any invariant subspace and any quotient of a homogeneous polynomial representation of $GL(V)$ is a homogeneous polynomial representation of the same degree.
   (d) If U and W are polynomial representations of $GL_K(V)$, then so are $U \oplus W$ and $U \otimes W$.
   (e) If U and W are homogeneous polynomial representations of $GL_K(V)$ of degrees p and q respectively, then $U \otimes W$ is a homogeneous polynomial representation of $GL(V)$ of degree $p + q$.
   (f) $g \mapsto \det^e g$ for e an integer $e \geq 0$ defines a homogeneous polynomial representation of $GL_K(V)$ of degree de.

(2) (Difficulty level 2) Show the following: K is infinite if and only if there is a unique polynomial (with respect to any fixed basis) that represents any fixed polynomial function on V.

(3) (Difficulty level 2) Suppose that K is finite of cardinality q. Determine the kernel of the surjective K-algebra morphism from $K[X_1, \ldots, X_d]$ to the ring $K[V]$ of polynomial functions on V. What is the dimension (as a K-vector space) of $K[V]$?

(4) (Difficulty level 2) Suppose that K is infinite. Show that the natural restriction map from polynomial functions on $\mathrm{End}_K(V)$ to functions on $GL_K(V)$ is an injection. (We may thus identify polynomial functions on $GL_K(V)$ with those on $\mathrm{End}_K V$.)

(5) (Difficulty level 2) Write a compact expression (as a rational function in t) for the formal power series $\sum_{n \geq 0} t^n \dim_K K[X_1, \ldots, X_d]_n$, where $K[X_1, \ldots, X_d]$ is the polynomial ring in d variables and $K[X_1, \ldots, X_d]_n$ is its subspace of polynomials of degree n.