Exercise set 2

Throughout F denotes a field and A a k-algebra. A ring is said to be *semisimple* if it a semisimple as a (left) module over itself.

(1) (difficulty level: 2) Show that any ring A with $1 \neq 0$ admits simple modules.

(2) (difficulty level: 2) Show that any module over a semisimple ring is semisimple.

(3) (Easy if you follow the hint) (Example of a non-semisimple finite dimensional complex algebra) Let B be the set of 2×2 upper triangular matrices with complex entries. With respect to standard matrix addition and multiplication, B is a $\mathbb{C}$-algebra. Let M be the set of 2×1 matrices with complex entries. Then with respect to multiplication from the left, M is a B module. Show that M is not semisimple. (Hint: Consider the submodule N consisting of 2×1 matrices with the $(2, 1)$ entry being 0. Show that N has no complement. In fact, show that N is the only non-trivial proper submodule of M.) Conclude that B is not semisimple.

(4) (difficulty level: 3) True or false?: If N is a semisimple submodule of a module M such that M/N is also semisimple, then M is semisimple.

(5) (difficulty level 3 without hint) A finite dimensional algebra over a field admits only finitely many isomorphism classes of simple modules. Solution: For any ring A, any simple module is a quotient of ${}_AA$: indeed, for any $m \neq 0$ in M, the homomorphism ${}_AA \to M$ by $a \mapsto am$ is onto. Now let A be an algebra of finite dimension over a field F. Fix a sequence ${}_AA = N_0 \supsetneq N_1 \supsetneq \ldots \supsetneq N_{k-1} \supsetneq N_k = 0$ of submodules of ${}_AA$ with all quotients N_j/N_{j+1} being simple. Such a sequence exists: indeed the length k of such a strictly decreasing sequence is bounded by $\dim_F A$. Given a simple A-module M, let $\varphi : A \twoheadrightarrow M$ be an A-module epimorphism, and let r be the least integer such that $\varphi|_{N_r} = 0$. Then $r \geq 1$ and $M \simeq N_{r-1}/N_r$. Thus any simple module is isomorphic to one of the quotients N_j/N_{j+1}.

Additional problems (optional)

(1) (Difficulty level 4 without hint) (Equivalence in general of the conditions in the definition of a semisimple module) Let M be a module such that any submodule of it admits a complement. Then M is a direct sum of (some of its) simple submodules. Solution: Fix a maximal collection $\mathfrak{C}$ of simple submodules of M whose sum is their direct sum: such a collection exists by Zorn. Let N be the sum of submodules in $\mathfrak{C}$, and suppose that $N \subsetneq M$. Choose $y \in M \setminus N$. Choose, by Zorn, a maximal proper submodule P containing N of $N + Ay$. Let S be a complement to P in $N + Ay$ (it exists because the hypothesis on M passes to submodules). Being isomorphic to $(N + Ay)/P$, it is simple. And its existence violates the maximality of $\mathfrak{C}$.

(2) (Difficulty level 3 without hint) Show that a ring that admits a finitely generated faithful[1] semisimple module is itself semisimple. Find a ring that is not semisimple but admits a faithful (necessarily non-finitely generated) semisimple module. (HInt: A commutative ring admits a faithful semisimple module iff its Jacobson radical is trivial.)

(3) (Difficulty level 3 without hint) (A version of Nakayama's lemma) Let M be a finitely generated A-module. Given a submodule N, there exists a maximal (proper) submodule of M containing N. (Hint: Zorn.) If M is finitely generated and non-zero, then there exists a primitive ideal $\mathfrak{a}$ such that $\mathfrak{a}M \subsetneq M$.[2] (Hint: Let N be a maximal proper submodule of M and take $\mathfrak{a}$ to be the annihilator of M/N.) Deduce the following: if A is a commutative local ring with maximal ideal $\mathfrak{m}$ and M a non-zero finitely generated A-module, then $\mathfrak{m}M \subsetneq M$.

[1] A module is faithful if no non-zero element of the ring kills the module.

[2] A ring is *primitive* if it admits a faithful simple module. A two sided ideal is *primitive* if the quotient by it is a primitive ring.