Exercise set 1

Throughout G denotes a group, and k a field.

(1) Check that the following two definitions of a k-algebra are equivalent:
   - R is a k-vector space and a ring with the underlying additive group of R in both cases being the same; the multiplication $R \times R \to R$ is k-bilinear.
   - R is a ring and there is a ring homomorphism $k \to R$ whose image lies in the centre of R.

(2) Check that the following two definitions of a k-linear representation V of a group G are equivalent:
   - $G \to GL(V)$ is a group homomorphism
   - V is a kG-module (in other words, there exists a k-algebra homomorphism from $kG \to \mathrm{End}_k(V)$)

(3) Let V be a vector space over k of finite dimension $n \geq 1$. Then $\mathrm{End}_k V$ is identified with the k-algebra $M_n(k)$ of $n \times n$ matrices over k.
   - For a subspace W, define $\ell_W := \{\varphi \in \mathrm{End}_k V \mid \varphi(W) = 0\}$. Show that ℓ_W is a left ideal in $\mathrm{End}_k V$, and moreover that every left ideal of $\mathrm{End}_k V$ is of the form ℓ_W for some W.
   - For a subspace W, define $\rho_W := \{\varphi \in \mathrm{End}_k V \mid \varphi(V) \subseteq W\}$. Show that ρ_W is a right ideal in $\mathrm{End}_k V$, and moreover that every right ideal of $\mathrm{End}_k V$ is of the form ρ_W for some W.
   - Show that $\mathrm{End}_k V$ is a simple k-algebra. (That is, it has precisely two two-sided ideals, namely, zero and itself.)

(4) A *multiplicative character* of a group G is a group homormorphism from G to the (multiplicative) group $k^\times$ of the non-zero elements in k. Show that $\sum_{g \in G} \xi(g) = 0$ for any non-trivial multiplicative character ξ of a finite group G. (A multiplicative character is called *trivial* if it is identically 1.)

(5) For a finite group G determine the centre of the group algebra kG. What is its dimension as a k-vector space?

(6) Observe that every multiplicative character of G factors through $G/(G,G)$. (Here (G,G) denotes the subgroup generated by the commutators $(g,h) := ghg^{-1}h^{-1}$, as g and h vary over all elements of G.)

(7) Observe the following: if G is a cyclic group of order n, then the group algebra $kG \simeq k[t]/(t^n - 1)$.

(8) Factorize the following determinant:

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_3 & x_4 & x_5 & \cdots & x_1 & x_2 \\ x_2 & x_3 & x_4 & \cdots & x_n & x_1 \end{vmatrix}$$