REPRESENTATION THEORY OF FINITE GROUPS

PROBLEMS SET 6

(1) Let X be a finite set with the action of a group G. Show that the multiplicity of the trivial representation in $\mathbf{C}[X]$ is $|G\backslash X|$, the number of G-orbits in X.
(2) Use the previous exercise, together with character theory to prove Burnside's lemma:
$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$
Here X^g denotes the set of points $x \in X$ such that $g \cdot x = x$.
(3) Let X, Y and Z be three G-sets. Given functions $k_1 : Y \times Z \to \mathbf{C}$ and $k_2 : X \times Y \to \mathbf{C}$, what is the function $k : X \times Z \to \mathbf{C}$ such that
$$T_k = T_{k_2} \circ T_{k_1}.$$
(4) The dihedral group D_{2n} acts on the set V of $2n$ vertices of the regular n-gon. What is $|G\backslash(V \times V)|$? What about $|G\backslash(V \times V \times V)|$?
(5) Each group G acts on itself by left multiplication. What are the relative positions of pairs of elements in G for this action?
(6) Let X_k denote the set of all subsets of order k in $\{1, \ldots, n\}$. Show that the representations $\mathbf{C}[X_k]$ and $\mathbf{C}[X_{n-k}]$ of S_n are isomorphic.

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