REPRESENTATION THEORY OF FINITE GROUPS

PROBLEMS SET 3

(1) Let A be an integer matrix with $\mathrm{RSK}(A) = (P, Q)$. Describe $\mathrm{RSK}(mA)$ for any positive integer m.

(2) Given non-negative integers a and b, describe the shape of the tableaux in $\mathrm{RSK}(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix})$. What about the tableaux corresponding to $\left(\begin{smallmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{smallmatrix}\right)$?

(3) Given a subset $S \subset \{1, \ldots, n\}$ of size k, where $k \leq n/2$, let A_S denote the $n \times 2$ matrix with entries defined by

$$a_{i1} = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{otherwise.} \end{cases}, \quad a_{i2} = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Note that every $n \times 2$ matrix with row sums all one, and column sums $(n-k, k)$ is of the form A_S for some such S. Supose that $\mathrm{RSK}(A_S) = (P_S, Q_S)$. Show that $\phi_{n,k} : S \mapsto Q_S$ is a bijection from the set of subsets of $\{1, \ldots, n\}$ of size k onto the set of all standard Young tableaux of shape $(n-l, l)$ for some $0 \leq l \leq k$. This gives a bijective proof of the identity

$$\binom{n}{k} = \sum_{l=0}^{k} f_{(n-k,k)},$$

for $k \leq n/2$ (see Problems Set 1, Ex. 3(b)).

(4) Enumerate the six subsets of size 2 in $\{1, 2, 3, 4\}$. Compute $\phi_{4,2}(S)$ for each of these subsets S.

(5) Give a direct construction of the bijection $\phi_{n,k}$ *without* using the RSK correspondence.

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