REPRESENTATION THEORY OF FINITE GROUPS

PROBLEMS SET 2

(1) Let T be a SSYT of shape μ and type λ. Let $\mu^{(i)}$ be the subshape of λ occupied by boxes containing numbers $1, \ldots, i$. By appending trailing zeroes (if necessary), write $\mu^{(i)}$ as a vector:

$$\mu^{(i)} = (\mu_1^{(i)}, \ldots, \mu_i^{(i)})$$

with i coordinates in decreasing order. The triangular array $(\mu_j^{(i)})$, $1 = 1, \ldots, l$, $j = 1, \ldots, i$ is called the *Gelfand-Tsetlin pattern* of T.

(a) Show that $\mu_j^{(i)} \geq \mu_j^{(i-1)} \geq \mu_{j+1}^{(i)}$ for all appropriate indices i and j.

(b) Express the type of T in terms of its Gelfand-Tsetlin pattern.

(2) Interpret our algorithm for constructing a SSYT of shape μ and type λ when $\mu \leq \lambda$ in terms of Gelfand-Tsetlin patterns. Extend the result about the existence of SSYT to Gelfand-Tsetlin patters with non-negative real coordinates.

(3) For each $0 \leq k \leq 4$, how many involutions does S_4 have with k fixed points? How many standard tableaux of of size 4 exist with k odd columns? Verify the second assertion about the RSK correspondence given in class.

(4) For arbitrary positive integer $k \leq n$, derive a formula for the number of involutions in S_n with k fixed points.

(5) Given three positive integers a, b, and c, find a formula for $M_{(a,b,c),(1,\ldots,1)}$, the number of $3 \times n$ matrices (where $n = a + b + c$) whose rows add up to a, b and c, and whose column sums are all 1.

(6) For any two elements x and y of a partially ordered set, $x \wedge y$ (the *greatest lower bound*, or the *meet* of x and y) is defined as the maximal element of the set

$$\{z \mid z \leq x \text{ and } z \leq y\},$$

provided that such an element exists, and is unique.

(a) What is $(i, j) \wedge (i', j')$ in the shadow partial order?

(b) Show that the shadow points of A are the maximal elements of the set

$$\{(i, j) \wedge (i', j') \mid (i, j) \text{ and } (i', j') \text{ are non-zero entries of } A\}.$$

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