REPRESENTATION THEORY OF FINITE GROUPS

PROBLEMS SET 11

(1) Compute $\chi_{(3,2,1)}(2,2,2)$.
(2) For each positive integer n, let $\lambda = (n, \ldots, n)$ denote the partition of n^2 with all parts equal to n. Show that
$$\chi_\lambda(2n-1, 2n-3, \ldots, 3, 1) = (-1)^{\lfloor n/2 \rfloor}.$$
(3) Let g_λ denote the number of standard Young tableaux of shape λ with 2 occuring in the *first column*. For any partition λ of $n \geq 2$, show that the character value $\chi_\lambda((1,2))$ at a 2-cycle $f_\lambda - 2g_\lambda$.
(4) Show that $\det \circ \rho_\lambda : S_n \to \mathbf{C}^*$ is the sign chatacter of S_n if and only if g_λ is odd. Such partitions are called *chiral partitions*[1] . [Hint: what are the eigenvalues of $\rho_\lambda(s_1)$?]
(5) Show that, if λ is a self-conjugate partition of n, then either a box can be added to the Young diagram of λ to obtain the Young diagram of a self-conjugate partition of $n+1$, or a box can be removed from the Young diagram of λ to obtain the Young diagram of a self-conjugate partition of $n+1$.
(6) We say that (λ, μ) is a self-conjugate cover if both λ and μ are self-conjugate, and the Young diagram of λ is obtained from the Young diagram of μ by removing one box. List all the self-conjugate covers involving partitions up to size eight.
(7) If (λ, μ) is a self-conjugate cover, $\lambda = \phi(\alpha)$, and $\mu = \phi(\beta)$ for some partitions α and β with distinct odd parts, then show that
$$\chi^\pm_\lambda(w_\alpha) = \pm\chi^\pm_\mu(w_\beta).$$

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[1]If n is an integer having binary expansion $n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}$, $\epsilon \in \{0,1\}$, $0 < k_1 < k_2 < \cdots < k_r$, the number of chiral partitions of n is $2^{k_2+\cdots+k_r}\left(2^{k_1-1} + \sum_{v=1}^{k_1-1} 2^{(v+1)(k_1-2)-\binom{v}{2}} + \epsilon 2^{\binom{k_1}{2}}\right)$; see Ayyer, Prasad, and Spallone, *JCTA*, vol. 150, 2017.