# The uses of Lie groups and Lie algebras in physics

## Groups and Symmetry

Static symmetry of objects in space $\longrightarrow$ transformation of variables which preserve given equations (ODE's, PDE's)

## Realisations and Representations of a group

$G$ : elements $g, g', a, b, \ldots$, identity $e$, inverses $g^{-1} \ldots$

Realisation : set $S$, maps $\phi_g$ of $S$ onto itself, one-to-one, onto, invertible :

$$\text{i)} \ \phi_{g'} \circ \phi_g = \phi_{g'g} \ ; \ \text{ii)} \ \phi_e = \text{Identity map};$$

$$\text{iii)} \ \phi_{g^{-1}} = \phi_g^{-1} \qquad (1)$$

Can require $S$ to be a differentiable manifold, $\phi_g$ diffeomorphism.

Representation $S \longrightarrow$ linear (real or complex) linear vector space $V$, $\phi$'s nonsingular linear transformations.

Dimension of $V$ : finite or infinite; Hilbert space case it has an inner product. Write $\mathcal{H}$ for $V$, $U(g)$ or $D(g)$ for $\phi_g$ : so in a (possibly unitary) representation :

$g \in G \to U(g)$ on $\mathcal{H}$,

i) $U(g')U(g) = U(g'g)$; ii) $U(e) = \mathbb{1}_{\mathcal{H}}$; iii) $U(g^{-1}) = U(g)^{-1}$;

iv) $U(g)^{\dagger} U(g) = \mathbb{1}_{\mathcal{H}}$ (2)

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<u>Dynamics and Symmetry in the classical case</u>

Properties of a physical system - real valued numerical dynamical variables. Choose an independent set of them.

Evolution of the system in time expressed by Equations of Motion (EOM)

Forms of EOM: Newtonian, Euler-Lagrange, Hamiltonian.

<u>Configuration space</u> $Q$, dimension $n$, differentiable manifold.

$q \in Q \to$ (local) coordinates $q^j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, independent (3)

From $Q$, can construct:

$Q$, dim $n$

- $TQ$ = Tangent bundle, dim. $2n$; local coordinates and velocities; $(q, \dot{q}) \in TQ$; E-L form of EOM
- $T^*Q$ = Cotangent bundle, dim. $2n$; positions $q^j$ and momenta $p_j$; $(q, p) \in T^*Q$; Hamiltonian form of EOM.

(4)

<u>State</u> at an instant of time : $(q, \dot{q}) \in TQ$, or $(q, p) \in T^*Q$.

<u>The E-L EOM</u> determined by $L(q, \dot{q}) \in \mathcal{F}(TQ)$, <u>Lagrangian</u>

$$\delta \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t)) = 0, \quad \delta q(t_1) = \delta q(t_2) = \delta t_1 = \delta t_2 = 0 \tag{5}$$

$$\Downarrow$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} = 0, \quad j = 1, 2, \cdots, n. \tag{6}$$

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<u>Properties of $T^*Q$</u> : It is a <u>symplectic manifold</u>

$\theta_0 = p_j \, dq^j =$ Canonical one-form on $T^*Q$

$\omega_0 = d\theta_0 = dp_j \wedge dq^j =$ canonical, symplectic, two-form on $T^*Q$,

$$d\omega_0 = 0, \quad \omega_0 \text{ nondegenerate} \tag{7}$$

$f, g \ldots \in \mathcal{F}(T^*Q)$ : $f(q,p)$, $g(q,p) \ldots$ <u>Poisson Bracket (PB)</u> :

$f, g \in \mathcal{F}(T^*Q) \rightarrow \{f, g\} \in \mathcal{F}(T^*Q)$,

$$\{f, g\}(q, p) = \frac{\partial f(q,p)}{\partial q^j} \frac{\partial g(q,p)}{\partial p_j} - \frac{\partial f(q,p)}{\partial p_j} \frac{\partial g(q,p)}{\partial q^j},$$

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0. \tag{8}$$

Smooth maps $\phi : T^*Q \rightarrow T^*Q$, one-to-one, onto, invertible,

which preserve above : <u>canonical transformations</u> or <u>symplectomorphisms</u>

$\Phi$ is a CT $\Leftrightarrow$ $\Phi^* \omega_0 = \omega_0 \Leftrightarrow \Phi^*\{f,g\} = \{\Phi^* f, \Phi^* g\}$ (9)

From $L(q,\dot q) \in \mathcal{F}(TQ)$ to hamiltonian $H(q,p) \in \mathcal{F}(T^*Q)$ by Legendre transformation:

Legendre map $TQ \to T^*Q$ : $p_j = \dfrac{\partial L(q,\dot q)}{\partial \dot q^j}$ ;

$$H(q,p) = p_j \dot q^j - L(q,\dot q). \qquad (10)$$

Phase space EoM:

$$\dot q^j = \{q^j, H\} = \frac{\partial H(q,p)}{\partial p_j}, \quad \dot p_j = \{p_j, H\} = -\frac{\partial H(q,p)}{\partial q^j},$$

$$\frac{d}{dt} f(q,p) = \{f, H\}(q,p). \qquad (11)$$

Solution of these EoM from $t_1$ to $t_2$ is a C.T., $H$ is its generator

Point transformation symmetries Lie group $G$ acting on $Q$ via diffeomorphisms $\phi$:

$$g \in G \to \phi_g : Q \to Q : q \in Q \to q' = \phi_g(q) = \phi(g;q) \in Q. \qquad (12)$$

Law for velocities:

$$\dot q^j \to \dot q'^j = \frac{\partial \phi^j(g;q)}{\partial q^k} \dot q^k \qquad (13)$$

$G$ is a group of symmetries for the system if

$$\mathcal{L}(q', \dot{q}') = \mathcal{L}(q, \dot{q}). \tag{14}$$

Then EOM are preserved, solutions mapped to solutions. Results:

i) $g \in G \rightarrow$ C.T.'s $\tilde{\phi}_g$ on $T^*Q$ determined by $\phi_g$, preserve $H$:

$\phi_g$ = point transformation on $Q \rightarrow \tilde{\phi}_g = CT$ on $T^*Q$,

$$\tilde{\phi}_g^* \omega_0 = \omega_0 \ , \quad \tilde{\phi}_g^* H = H. \tag{15}$$

ii) These CT's preserve the EOM (11)

iii) Elements $g$ near $e$ act via infinitesimal CT's:

$\{e_r\}$ = basis for $\underline{G}$, $\alpha^r e_r \in \underline{G}$, $\alpha^r \in \mathbb{R}$: $e_r \rightarrow J_r(q,p) \in \mathcal{F}(T^*Q)$,

$$|\epsilon| \ll 1: \ \delta q^j = \epsilon\{q^j, \alpha^r J_r\}, \ \delta p_j = \epsilon\{p_j, \alpha^r J_r\}. \tag{16}$$

iv) $J_r(q,p)$ determined up to additive constants, are constants of motion (COM):

$$\frac{dJ_r}{dt} = \{J_r, H\} = 0. \tag{17}$$

v) $$\{J_r, J_s\} = C_{rs}^{\ \ t} J_t + d_{rs},$$

$$C_{rs}^{\ \ t} = \text{structure constants}, \quad d_{rs} = \text{constants} \tag{18}$$

All this is 'Noether's Theorem!

Basic fact: Lie group of symmetries acts via CT's, generators are COM.

Point transformations: $J_r(q,p)$ linear in momenta.

Non Noether symmetries: $J_r(q,p)$ not linear in momenta.

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## Dynamics and Symmetry in Quantum Mechanics

Given QM system: complex Hilbert space $\mathcal{H}$, finite or infinite dimensional, vectors $\psi, \phi, \psi', \phi', \ldots$, inner product $(\phi, \psi)$ or $\langle\phi|\psi\rangle$.

$$\psi \in \mathcal{H}, \ \langle\psi|\psi\rangle = \|\psi\|^2 = 1: \ e^{i\alpha}\psi, \ 0 \le \alpha \le 2\pi \longrightarrow \text{Same pure state} \quad (19)$$

Superposition Principle:

$$\psi = c_1\psi_1 + c_2\psi_2 + \cdots \longrightarrow \text{new pure state.} \quad (20)$$

Typically, $\mathcal{H}$ carries irreducible representation of 'primitive' set of operators obeying given commutation or other algebraic relations.

Dynamical variables are hermitian operators $\hat{A}, \hat{B} \ldots$ on $\mathcal{H}$.

The EOM of QM:

$$i\hbar \frac{d}{dt}\psi(t) = \hat{H}(t)\psi(t), \quad \hat{H} = \text{hamiltonian operator} \quad (21)$$

A Lie group $G$ represented unitarily on $\mathcal{H}$ is a symmetry group if $\hat{H}$ is invariant:

$$G \text{ is a Lie group of symmetries} \iff \hat{H}\, U(g) = U(g)\, \hat{H}, \quad g \in G. \qquad (22)$$

Results: i) $\psi(t)$ obeys (21), $\psi'(t) = U(g)\,\psi(t) \Rightarrow \psi'(t)$ obeys (21). (23)

ii) $g$ near $e \in G$: $U(g) \simeq \mathbb{1} - i\alpha^r \hat{J}_r + \cdots$,

$$\hat{J}_r^{\dagger} = \hat{J}_r, \quad [\hat{J}_r, \hat{J}_s] = i\, C_{rs}{}^{t}\, \hat{J}_t,$$

$$[\hat{J}_r, \hat{H}] = 0. \qquad (24)$$

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## Wigner's theorem in QM

$\psi, \phi \in \mathcal{H}$, $\|\psi\| = \|\phi\| = 1$ :

$$\text{transition probability } (\psi \to \phi) = |(\phi, \psi)|^2. \qquad (25)$$

$\hat{P}(\psi) = \psi\psi^{\dagger}$ or $|\psi\rangle\langle\psi|$, $\hat{P}(\phi) = \phi\phi^{\dagger}$ or $|\phi\rangle\langle\phi|$: one dim. projections:

$$\text{Prob. } (\psi \to \phi) = \mathrm{Tr}\,(\hat{P}(\phi)\,\hat{P}(\psi)) \qquad (26)$$

Wigner symmetry is one-to-one, onto, invertible map $\Omega$:

$\Omega(\hat{P}(\psi)) = \hat{P}(\psi')$, $\Omega(\hat{P}(\phi)) = \hat{P}(\phi')$, ...; $\psi', \phi', \ldots$ determined up to phases.

$$|(\phi', \psi')|^2 = \mathrm{Tr}\,(\hat{P}(\phi')\,\hat{P}(\psi')) = \mathrm{Tr}\,(\hat{P}(\phi)\,\hat{P}(\psi)) = |(\phi, \psi)|^2. \qquad (27)$$

Theorem $\Omega$ is a symmetry operation $\Rightarrow \Omega(\hat{\rho}(\psi)) = \hat{\rho}(U\psi)$,

$U$ = map $\mathcal{H} \to \mathcal{H}$, either linear unitary or antilinear antiunitary (28)

Limit to unitary case. For Lie group of symmetries, consistency demands:

$$g \in G \to U(g) \text{ on } \mathcal{H}, \quad U(g)^{\dagger} U(g) = \mathbb{1},$$

$$U(g')\,U(g) = e^{i\omega(g',g)}\, U(g'g), \quad \omega(g',g) = \text{real phase}. \tag{29}$$

If $\omega \neq 0$: unitary ray representation of $G$; if $\omega = 0$: true or vector representation. In general: generators obey

$$[\hat{J}_r, \hat{J}_s] = i\, C_{rs}^{\ \ t}\, \hat{J}_t + i d_{rs}. \tag{30}$$

Symmetry groups related to space time

a) Three dimensional proper orthogonal rotation $SO(3)$: isotropy of space:

$$SO(3) = \{ A = 3\times 3 \text{ real matrix} \mid A^T A = \mathbb{1},\ \det A = 1 \}. \tag{31}$$

Compact, order 3, rank 1. In any unitary representation (UR):

$$[\hat{J}_r, \hat{J}_s] = i\, \epsilon_{rst}\, \hat{J}_t, \quad r,s,t = 1,2,3. \tag{32}$$

In QM, need to also use

$$SU(2) = \{ u = 2\times 2 \text{ complex matrix} \mid u^{\dagger} u = \mathbb{1},\ \det u = 1 \}. \tag{33}$$

Rotational invariance in QM uses UR's and UIR's of $SO(3)$.

b) Euclidean group in three dimensions, $E(3)$ homogeneity and isotropy of space: order 6, semidirect product.

$$E(3) = \{ g = (A, \underline{a}) \mid A \in SO(3), \underline{a} \in \mathbb{R}^3 \}; \qquad (a)$$

$$g \in E(3): \ \underline{x} \in \mathbb{R}^3 \to \underline{x}' = g\,\underline{x} = A\underline{x} + \underline{a} \in \mathbb{R}^3; \qquad (b)$$

$$g'g = (A', \underline{a}')(A, \underline{a}) = (A'A, \ \underline{a}' + A'\underline{a}). \qquad (c) \qquad (34)$$

Generators $\hat{\underline{J}}$ and $\hat{\underline{P}}$:

$$[\hat{J}_r, \hat{J}_s] = i\,\epsilon_{rst}\,\hat{J}_t, \quad [\hat{J}_r, \hat{P}_s] = i\,\epsilon_{rst}\,\hat{P}_t, \quad [\hat{P}_r, \hat{P}_s] = 0. \qquad (35)$$

c) Galilei group $\mathcal{G}$ Ten parameter group, indicate structure.

$$A \in SO(3), \ \underline{a} \in \mathbb{R}^3, \ b \in \mathbb{R}, \ \underline{v} \in \mathbb{R}^3 : g = (A, \underline{v}, b, \underline{a}):$$

$$(\underline{x}, t) \to (\underline{x}', t') = g\,(\underline{x}, t) = (A\underline{x} + \underline{a} + \underline{v}\,t, \ t + b);$$

$$\mathcal{G} = \{ g = (A, \underline{v}, b, \underline{a}) \mid A \in SO(3), \underline{v} \in \mathbb{R}^3, b \in \mathbb{R}, \underline{a} \in \mathbb{R}^3 \};$$

$$g'g = (A', \underline{v}', b', \underline{a}')(A, \underline{v}, b, \underline{a}) = (A'A, \underline{v}' + A'\underline{v}, b' + b, \underline{a}' + A'\underline{a} + b\,\underline{v}') \quad (36)$$

d) Poincaré group $\mathcal{P}$ basic to special relativity. Space time is $\mathbb{R}^4$ plus pseudo-Euclidean, ie Minkowski, metric.

$$x = (x^\mu) = (x^0 = ct,\ x^1, x^2, x^3),\quad \mu = 0,1,2,3;$$

$$x \cdot x = \eta_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2,$$

$$\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1). \tag{37}$$

$\mathcal{P}$ - semidirect product of space time translations and homogeneous Lorentz transformations $SO(3,1)$:

$$SO(3,1) = \{ \Lambda = (\Lambda^\mu{}_\nu) = 4\times 4 \text{ real matrix} \mid \Lambda^T \eta \Lambda = \eta,\ \Lambda^0{}_0 \geq 1,\ \det\Lambda = 1 \};$$

$$\text{Translations} = \{ a^\mu \in \mathbb{R}^4 \};$$

$$(\Lambda, a^\mu) \in \mathcal{P}:\quad x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu;$$

$$(\Lambda', a'^\mu)(\Lambda, a^\mu) = (\Lambda'\Lambda,\ a'^\mu + \Lambda'^\mu{}_\nu a^\nu). \tag{38}$$

Wigner's classic paper on UIR's of $\mathcal{P}$: 1939

The groups $Sp(2n, \mathbb{R})$  Basic Cartesian variables, operators:

$$Q = \mathbb{R}^n,\quad T^*Q = T^*\mathbb{R}^n \simeq \mathbb{R}^{2n},\quad \mathcal{H} = L^2(\mathbb{R}^n).\ \text{Heisenberg CCR's:}$$

$$\hat{q}_j^\dagger = \hat{q}_j,\ \hat{p}_j^\dagger = \hat{p}_j;\ [\hat{q}_j, \hat{p}_k] = i\delta_{jk},\ [\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0,\ j,k = 1,2,\dots,n. \tag{39}$$

Concise form:

$$\hat{\xi} = (\hat{\xi}_a),\ a = 1,2,\dots,2n:\quad \hat{\xi}_j = \hat{q}_j,\quad \hat{\xi}_{n+j} = \hat{p}_j. \tag{40}$$

CCR's: $\hat{\xi}_a^\dagger = \hat{\xi}_a$, $[\hat{\xi}_a, \hat{\xi}_b] = i\beta_{ab}$, $\beta = \begin{pmatrix} 0 & 1_{n\times n} \\ -1_{n\times n} & 0 \end{pmatrix}$. (41)

Group of linear canonical transformation:

$\hat{\xi}_a' = S_{ab}\,\hat{\xi}_b$, $S = (S_{ab}) = 2n\times 2n$ real matrix,

$$S \beta S^T = \beta. \quad (42)$$

$$Sp(2n,\mathbb{R}) = \{ S = \text{real } 2n\times 2n \text{ matrix} \,/\, S\beta S^T = \beta \}. \quad (43)$$

$S \in Sp(2n,\mathbb{R})$: $\hat{\xi}_a' = S_{ab}\,\hat{\xi}_b = U(S)\,\hat{\xi}_a\,U(S)^{-1}$. (44)

$$U(S')\,U(S) = \pm\, U(S'S)$$

So $\{U(S)\}$ give a two-valued UR of $Sp(2n,\mathbb{R})$, or a true UR of $Mp(2n)$, double cover of $Sp(2n,\mathbb{R})$, the metaplectic group discovered by A. Weil. This UR is direct sum of two UIR's.

Very useful in quantum mechanics, classical optics, and quantum optics.