TUTORIAL ON SERRE'S THEOREM AND AUTOMORPHISMS OF LIE ALGEBRAS

For first five problems, the notation is as in §18 of *Humphreys: Introduction to Lie algebras and representation theory*.

1. Verify that V is indeed a representation of the Lie algebra L_0.

2. Using this representation, prove that the subalgebra X (respectively Y) generated by $x_1, \ldots, x_l$ (respectively by $y_1, \ldots, y_l$) is free.

3. Prove that the ideal K is contained in every ideal of L_0 having finite comension.

4. Prove that an inclusion of Dynkin diagrams induces a natural inclusion of the corresponding semisimple Lie algebras.

5. Understand the proof of Theorem 18.3.

6. Let L be a Lie Algebra. Denote $\mathrm{Aut}(L)$ for the set of all automorphisms of L.
 (1) For $x \in L$ such that $ad(x)$ is nilpotent show that $exp(ad(x)) \in \mathrm{Aut}(L)$.
 (2) Show that the group of inner automorphisms $Int(L) :=< exp(ad(x)) \mid ad(x)\ nilpotent >$ is a normal subgroup of $\mathrm{Aut}(L)$.
 (3) Suppose $L \subset gl(V)$ and $x \in L$ is nilpotent. Then $exp(ad(x))(y) = exp(x).y.exp(x)^{-1}$ for all $y \in L$.
 (4) For $L = sl_2$, calculate explicitely $\sigma = exp(ad(x)).exp(ad(-y)).exp(ad(x))$. Further show that $\sigma\colon L \to L$ is given by $z \mapsto szs^{-1}$ where $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

7. Let L be a semisimple Lie algebra with generators $\{x_i, y_i, h_i\}$. Show that the set map $x_i \mapsto -y_i$, $y_i \mapsto -x_i, h \mapsto -h, \forall i$ lifts to an automorphism σ of the Lie algebra L. It is an involution. This element is the longest element in the Weyl group of L. (Hint: Use Serre's Theorem.)

8. Let Φ be an abstract root system (reduced). Show that $\mathrm{Aut}(\Phi) \cong W \rtimes \Gamma$ where W is the Weyl group and Γ is graph automorphisms.

9. Let L be a semisimple Lie algebra and Φ corresponding root system.
 (1) Show that $W \leq \mathrm{Aut}(L)$.
 (2) Show that $\mathrm{Aut}(\Phi) \leq \mathrm{Aut}(L)$.
 (3) What is $\mathrm{Aut}(L)$?

10. Let L be a semisimple Lie algebra. Consider the ad map. Find out highest weight and a highest weight vector?