AIS workshop on Lie algebra
Tutorial

(1) (a) Prove that any polynomial in n variables over a field of F characteristic 0 is a linear combination of powers of linear polynomials. (Hint: Use induction on n. Expand $(x_1 + ax_2)^k$ and then use a Vandermonde determinant argument to show that k-th powers of linear polynomials span a space of correct dimension when $n = 2$).

(b) Conclude that for a semi-simple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the set $\{\lambda^k : \lambda \in \Lambda, k \in \mathbb{Z}_{>0}\}$ spans $F[\mathfrak{h}]$.

(2) Prove that the weight lattice Λ is Zariski dense in $\mathfrak{h}^*$. (Hint: Identify Λ with $\mathbb{Z}^n$. Then its like proving $\mathbb{Z}^n$ is Zariski-dense in $\mathbb{C}^n$ by using induction on n).

(3) Prove that in a semi-simple Lie algebra $\mathfrak{g}$, the set of (regular) semi-simple elements form a Zariski dense subset of $\mathfrak{g}$. (Hints: Its like proving the set of $n \times n$ diagonalizable matrices is dense in the set of all $n \times n$ matrices. Use discrminant of characteristic polynomials).

(4) Let $\lambda_1, \lambda_2 \in \mathfrak{h}^*$ be two elements whose W orbits are disjoint. Then there exists a W-invariant polynomial $f \in \mathbb{C}[\mathfrak{h}^*]$ such that $f(\lambda_1) = 1$ and $f(\lambda_2) = 0$. (Hints: First try to find an element $f \in \mathbb{C}[\mathfrak{h}^*]$ having the above properties. Then normalize it over the Weyl group elements to get an invariant polynomial having the same properties).

(5) Let $G = SL_2$. Let $\mathfrak{g}$ be the Lie algebra of G. Compute $\mathbb{C}[\mathfrak{h}]^W, C[\mathfrak{g}]^G$ and $Z(U(\mathfrak{g}))$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. (Hints: All the three algebras are isomorphic by Chevalley's restriction theorem and Hirish-Chandra's theorem and they are all generated by one element. Find the respective generator for each of them).

(6) Let $V = \mathbb{R}^n$. Then the symmetric group S_n acts on V by permuting the coordinates. Find a minimal generating set for $\mathbb{R}[V]^{S_n}$ and show that the product of degrees of the elements of the minimal generating set is $|S_n| = n!$. (Hint: Every symmetric polynomial can be written as a polynomial in elementary symmetric polynomials).

(7) Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. Show that $\rho - w(\rho) = \sum_{\alpha\in\Phi^+(w^{-1})} \alpha$, where $\Phi^+(w^{-1}) = \{\alpha \in \Phi^+ : w^{-1}(\alpha) < 0\}$. Then conclude that ρ is the sum of fundamental weights. (Hints: After proving the 1st part using induction on $l(w)$, apply simple reflections and compare the co-efficients).

(8) Compute $\mathfrak{p}(\alpha_0)$, where $\mathfrak{p}$ is the Kostant partition function and α_0 is the highest root for $\mathfrak{sl}_6$. (By definition $\mathfrak{p}(\lambda)$= the number of ways λ can be written as sum of positive roots.) (Hint: The highest root in this case is $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. What are all positive roots ?).