AFFINE LIE ALGEBRAS - PROBLEM SET 3

All Lie algebras are over $\mathbb{C}$.

(1) Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (or more generally a Kac-Moody algebra). Let X be the Dynkin diagram of $\mathfrak{g}$ and let $\alpha_i : i = 1 \cdots l$ be the simple roots of $\mathfrak{g}$. Recall that the simple roots correspond to the vertices of X. Given an element $\alpha \in \mathfrak{h}^*$, write $\alpha := \sum_{i=1}^{l} c_i \alpha_i$. We define its support by $\operatorname{supp} \alpha := \{i : c_i \neq 0\}$. Prove that if α is a root, then its support is a connected subdiagram of X.

(2) Recall the construction of untwisted affine Lie algebras as central extension of loop algebras. Prove using this that the center is $\mathbb{C}K$ for such an algebra.

(3) Let R be a $\mathbb{C}$-algebra. Let $\mathfrak{sl}_n(R)$ denote the set of $n \times n$ matrices with entries in R and trace 0. Prove that $\mathfrak{sl}_n(R)$ is a complex Lie algebra, and that it is isomorphic as Lie algebras to $\mathfrak{sl}_n(\mathbb{C}) \otimes_{\mathbb{C}} R$.

Definition: The Heisenberg or oscillator algebra is the Lie algebra $\mathfrak{O}$ with basis $\{a_n : n \in \mathbb{Z}\} \cup \{K\}$, with the Lie bracket defined by $[K, a_n] = 0, [a_n, a_m] = n\delta_{n+m,0}K$ for all $n, m \in \mathbb{Z}$.

(4) Let $\widehat{\mathfrak{sl}_2}$ denote the *affinization* of $\mathfrak{g} := \mathfrak{sl}_2$, i.e

$$\widehat{\mathfrak{sl}_2} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} + \mathbb{C}K + \mathbb{C}d$$

Construct (at least) two different Lie subalgebras of $\widehat{\mathfrak{sl}_2}$ which are isomorphic to $\mathfrak{O}$.

(5) Find all elements of the following Lie algebras
  (a) $\operatorname{Der}(\mathbb{C}[t, t^{-1}])$.
  (b) $\mathfrak{d} := \{d \in \operatorname{Der} \mathfrak{O} : d(K) = 0\}$.