AFFINE LIE ALGEBRAS - PROBLEM SET 2

All Lie algebras are over $\mathbb{C}$.

(1) Let $n \geq l$; suppose we have a $(2n-l)\times(2n-l)$ symmetric matrix with a block decomposition
$$\begin{bmatrix} A & B \\ {}^tB & 0 \end{bmatrix}$$
where A is an $n \times n$ symmetric matrix and B is an $n \times (n-l)$ matrix. Show that if $\mathrm{rank} A = l$ and $\mathrm{rank}(A \mid B) = n$, then the matrix is invertible. Is the converse true ?

(2) If R is a commutative $\mathbb{C}$-algebra and $\mathfrak{g}$ a Lie algebra, show that $\mathrm{Der}(R \otimes \mathfrak{g}) = \mathrm{Der}(R) \otimes \mathrm{id}_{\mathfrak{g}} + R \otimes \mathrm{Der}(\mathfrak{g})$.

(3) With notation as in Monday's last lecture: Let $X_l^{(1)}$ denote the extended Cartan matrix of a finite dimensional semisimple Lie algebra $\mathring{\mathfrak{g}}$. Let $\mathfrak{h} := \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$, $\Pi = \{\alpha_0, \cdots, \alpha_l\}$, $\Pi^\vee = \{\alpha_0^\vee, \cdots, \alpha_l^\vee\}$. Here, $\alpha_i \in \mathring{\mathfrak{h}}^*$ (for $i = 1 \cdots l$) is extended to $\mathfrak{h}^*$ by zero on K and d; $\alpha_0 := \delta - \theta$ where $\delta(\mathring{\mathfrak{h}}) = 0, \delta(K) = 0, \delta(d) = 1$; $\alpha_i^\vee (i = 1 \cdots l) \in \mathring{\mathfrak{h}} \subset \mathfrak{h}$; $\alpha_0^\vee = -\theta^\vee + K$. Now, check that $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of the matrix $X_l^{(1)}$.

(4) Let $(V, (\cdot \mid \cdot))$ be a real inner product space and let α_i, $i = 1 \cdots n$ be **any non-zero vectors** (not necessarily linearly independent) in V. Define the $n \times n$ matrix $A := [a_{ij}]$ where $a_{ij} := \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$. Show that A is symmetrizable. Use this to show that the Cartan matrices and the extended Cartan matrices associated to finite dimensional simple Lie algebras are symmetrizable.

(5) Let $\mathfrak{g}$ be one of the Lie algebras $\mathfrak{sl}_2$ or $\mathfrak{sl}_3$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and $l := \dim(\mathfrak{h})$. Let $(\cdot \mid \cdot)$ be a nondegenerate invariant form on $\mathfrak{g}$ (for eg, the Killing form). Let $\alpha_i \in \mathfrak{h}^*$ $(i = 1 \cdots l)$ denote the simple roots of $\mathfrak{g}$. Find all nonzero vectors v in $\mathfrak{h}^*$ such that setting $\alpha_{l+1} := v$ makes the $(l+1)\times(l+1)$ matrix $A := \left[\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}\right]_{i,j=1}^{l+1}$ a **generalized Cartan matrix**.