AFFINE LIE ALGEBRAS - PROBLEMS

All Lie algebras are over $\mathbb{C}$.

(1) Let $\mathfrak{g}$ be a Lie algebra. A *2-cocycle* is a bilinear map $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ satisfying : (i) $\psi(X,Y) = -\psi(Y,X)$ for all $X, Y \in \mathfrak{g}$ and (ii) $\psi([X,Y],Z)+\psi([Y,Z],X)+\psi([Z,X],Y) = 0$ for all $X, Y, Z \in \mathfrak{g}$.

A *2-coboundary* is a bilinear map $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ such that $\psi(X,Y) = f([X,Y])$ for some $f \in \mathfrak{g}^*$.

(a) Let $B^2(\mathfrak{g})$ denote the space of 2-coboundaries, and $Z^2(\mathfrak{g})$ denote the space of 2-cocyles. Prove that $B^2(\mathfrak{g}) \subset Z^2(\mathfrak{g})$. Define $H^2(\mathfrak{g}) := Z^2(\mathfrak{g})/B^2(\mathfrak{g})$.

(b) **Definition:** A central extension of $\mathfrak{g}$ is an exact sequence of Lie algebras

$$0 \to \mathfrak{k} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where $\mathfrak{k}$ is contained in the center of $\tilde{\mathfrak{g}}$.

Now, let $(\mathfrak{g}, [\cdot,\cdot]')$ be a Lie algebra. Given a bilinear $\mathbb{C}$-valued map ψ on $\mathfrak{g}$, let $\mathfrak{k}$ be a one dimensional complex vector space $\mathbb{C}K$. Let $\tilde{\mathfrak{g}} := \mathbb{C}K \oplus \mathfrak{g}$, and define a bracket $[\cdot,\cdot]$ on this via the relations:

$$[K,K] = 0, [K,X] = 0, [X,Y] = [X,Y]' + \psi(X,Y)K \quad \forall X, Y \in \mathfrak{g}$$

Prove that $[\cdot,\cdot]$ is a Lie bracket $\iff \psi \in Z^2(\mathfrak{g})$.

(c) **Definition:** The central extensions $0 \to \mathfrak{k}_1 \to \tilde{\mathfrak{g}}_1 \to \mathfrak{g} \to 0$ and $0 \to \mathfrak{k}_2 \to \tilde{\mathfrak{g}}_2 \to \mathfrak{g} \to 0$ are said to be isomorphic if there exists an isomorphism of Lie algebras $\phi : \tilde{\mathfrak{g}}_1 \to \tilde{\mathfrak{g}}_2$ such that $\phi(\mathfrak{k}_1) = \mathfrak{k}_2$ and the induced quotient map $\mathfrak{g} \to \mathfrak{g}$ is the identity.

Prove that the one-dimensional central extensions of $\mathfrak{g}$ defined by two 2-cocyles ψ_1 and ψ_2 are isomorphic $\iff \psi_1 - \psi_2 \in B^2(\mathfrak{g})$. Thus, the one dimensional central extensions of $\mathfrak{g}$ are in 1-1 correspondence with $H^2(\mathfrak{g})$.

(d) Let $0 \neq \psi_1 \in H^2(\mathfrak{g})$. Let $c \in \mathbb{C}$ be a non-zero scalar, and define $\psi_2 := c\psi_1$. Show that the one dimensional central extensions of $\mathfrak{g}$ defined by ψ_1 and ψ_2 are not isomorphic as central extensions, but are isomorphic as Lie algebras (i.e the middle terms in the exact sequences are isomorphic).

(2) Let $\mathfrak{g}$ and $\mathfrak{d}$ be given Lie algebras, together with a homomorphism of Lie algebras $\phi : \mathfrak{d} \to \operatorname{Der}\mathfrak{g}$. The semidirect product $\mathfrak{g} \rtimes \mathfrak{d}$ is the vector space $\mathfrak{g} \oplus \mathfrak{d}$ with bracket

$$[X+d, Y+e] := ([X,Y]_{\mathfrak{g}} + \phi(d)(Y) - \phi(e)(X)) + [d,e]_{\mathfrak{d}}$$

Prove that this is a Lie bracket.

(3) Let M be an abelian group. A Lie algebra $\mathfrak{g}$ is said to be M-graded if it admits a decomposition as a direct sum of subspaces $\mathfrak{g} = \oplus_{j \in M} \mathfrak{g}_j$, such that $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$ for all $j, k \in M$. Given a homomorphism $f : M \to (\mathbb{C}, +)$, define $d : \mathfrak{g} \to \mathfrak{g}$ by $d(X) = f(j)X$ for all $X \in \mathfrak{g}_j$, $j \in M$. Prove that d is a derivation of $\mathfrak{g}$.