Exercises on Lie Algebra

(Representations of Lie Algebras)

Notation : F is an algebraically closed field of characteristic 0.

Exercise 1. Let $\lambda \in F$. Let $Z(\lambda)$ be a vector space over F with countable basis $\{v_0, v_1, v_2, \ldots\}$. Define the action of $\mathfrak{sl}(2)$ by formulas:

$$hv_i = (\lambda - 2i)v_i, yv_i = (i+1)v_{i+1}, xv_i = (\lambda - i + 1)v_{i-1}.$$

We set $v_{-1} = 0$. Then prove the following,

(1) The space $Z(\lambda)$ is an $\mathfrak{sl}(2)$-module and every proper submodule contains at least one maximal vector.
(2) $Z(\lambda)$ is an irreducible representation if and only if $\lambda + 1$ is not a nonnegative integer.
(3) For r a nonnegative integer define a map $\phi \colon Z(\mu) \to Z(\lambda)$ by $v_i \mapsto v_{i+r}$ where $\mu = \lambda - 2r$. Then ϕ is an injective $\mathfrak{sl}(2)$-module homomorphism.
(4) In the case $\lambda + 1 = r$ the $Im(\phi)$ and $V(\lambda) = Z(\lambda)/Im(\phi)$ are irreducible.
(5) $Z(\lambda)$ is not always completely reducible.
(6) $V(\lambda)$ is finite dimensional if and only if λ is a nonnegative integer.

Exercise 2. Let V and W be representations of L. Show that $V^*, V \otimes W, V^{\otimes n}, S^n(V), \wedge^n(V)$ are again representations.

Exercise 3. Let L be a semisimple Lie algebra. Fix a cartan subalgebra H and let Φ be root system. Fix simple roots Δ. Show that

$$B = H \oplus \oplus_{\alpha \in \Phi^+} L_\alpha$$

is a Borel subalgeba.

Exercise 4. Let $\phi \colon L \to \mathfrak{gl}(V)$ be a finite dimensional representation. Then there exists a maximal vector of weight λ for some $\lambda \in H^*$.

Exercise 5. Check the following identities in $U(L)$ for $k \geq 0, 1 \leq i, j \leq l$:
(1) $[x_j, y_i^{k+1}] = 0$ when $i \neq j$.
(2) $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$
(3) $[x_i, y_i^{k+1}] = -(k+1)y_i^k(k.1 - h_i)$.

Exercise 6. Draw the diagram of root lattice and weight lattice in the case of 2-dimensional root systems.

Exercise 7. Take the standrad 2-dimensional representation $V =< X, Y >$ of sl_2. Find out the representations $sym^n(V)$.

Exercise 8. Do the exercise 20.2 from Humphreys and as many as you can from section 21 (6,7,8,10,11).

Exercise 9. Read section 2.3 and the last paragraph of section 7.2 from Humphreys.