KNR TUTORIAL SHEET 3
CASIMIR, COMPLETE REDUCIBILITY, AND PRESERVATION OF JORDAN DECOMPOSITION

Humphreys 6.1, 6.6, 6.7, 6.8, 6.9, 6.4, 6.5.

(1) (Complete reducibility for $\mathrm{sl}_2(\mathbb{C})$) Let V be a finite dimensional $\mathrm{sl}_2(\mathbb{C})$-module.
- Show that the "Casimir element" $C = XY + YX + \frac{1}{2}H^2$ belongs to the centre of the universal enveloping algebra of $\mathrm{sl}_2(\mathbb{C})$. Thus the generalized eigenspaces of C are $\mathrm{sl}_2(\mathbb{C})$-invariant.
- A vector v of V is *primitive* if it is an eigenvector for H and Xv vanishes modulo a proper invariant subspace W of V. Show that V is generated as an $\mathrm{sl}_2(\mathbb{C})$-module by its primitive vectors.
- If v is a primitive vector with weight λ and belongs to the generalized eigenspace of C with eigenvalue μ, then $\frac{1}{2}\lambda^2 + \lambda = \mu$.
- If all the primitive vectors of V have the same weight, then they are all highest weight vectors.
- Conclude that all the primitive vectors belonging to a generalized C-eigenspace of V have the same weight and therefore are all highest weight vectors. Thus each such generalized eigenspace is generated by highest weight vectors. But, as we have seen, a finite dimensional module generated by a highest weight vector is semisimple.