KNR TUTORIAL SHEET 2
FINITE DIMENSIONAL REPRESENTATION THEORY OF $\mathrm{sl}_2(\mathbb{C})$

(1) Show that $\mathrm{sl}_2(\mathbb{C})$ is a simple Lie algebra.

(2) Let V be a finite dimensional $\mathrm{sl}_2(k)$-module, where k is a field of characteristic 0. Show that the elements X and Y act nilpotently on V (directly, without invoking the "preservation of Jordan decomposition"). Show that neither the assumption on the characteristic of the field nor that of the finite dimensionality of V can be omitted.

(3) For v a highest weight vector for $\mathrm{sl}_2(\mathbb{C})$ of weight λ,

$$XY^{(n)}v = (\lambda - n + 1)Y^{(n-1)}v.$$

(4) Use Lie's theorem to prove the existence of a highest weight vector in an arbitrary finite dimensional $\mathrm{sl}_2(\mathbb{C})$-module. (Hint: Consider the Lie algebra spanned by X and H.)

(5) Consider $\mathrm{sl}_2(\mathbb{C})$ imbedded in $\mathrm{sl}(3, \mathbb{C})$ in the upper left corner. Decompose into irreducibles $\mathrm{sl}(3, \mathbb{C})$ under the adjoint action of this subalgebra.

(6) Decompose $V_m \otimes V_n$ into irreducibles, where V_m and V_n are the irreducible modules of $\mathrm{sl}_2(\mathbb{C})$ corresponding to highest weights m and n respectively.