KNR TUTORIAL SHEET 1
AUTOMORPHISMS AND DERIVATIONS

(1) The group $\mathrm{Aut}(V)$ of algebra automorphisms of a finite dimensional algebra V is a Zariski closed subgroup of $\mathrm{GL}(V)$.
(2) Compute the tangent spaces to the groups $\mathrm{SL}(V)$, $\mathrm{SO}(V)$, and $\mathrm{Sp}(V)$. What are their dimensions?
(3) Let V be an algebra over a field of characteristic 0 and D a derivation of V. Then $D^{(n)}(v \cdot w) = \sum_{0 \leq k, l \leq n; k+l \leq n} D^{(k)}(v) \cdot D^{(l)}(w)$.
(4) The *inner derivations* of a Lie algebra $\mathfrak{g}$ are those of the form $\mathrm{ad}\, X$ for some X in $\mathfrak{g}$. The space of inner derivations is an ideal in the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of all derivations. In fact, $[D, \mathrm{ad}\, X] = \mathrm{ad}\, DX$.
(5) Recall Jordan decomposition.
(6) The semisimple and nilpotent parts of a derivation (of a finite dimensional algebra) are themselves derivations. (Hint: Pass to an algebraic closure and think of generalized eigenspaces.)
(7) Let V be a finite dimensional algebra. The space $\mathrm{Der}(V)$ of derivations of V is invariant for the adjoint action of $\mathrm{Aut}(V)$. Suppose that D is a nilpotent derivation and the characteristic is 0 of the underlying field. Then $\exp(D)$ is an automorphism of V (as we've seen in class). Show that $\mathrm{ad}\, D$ acts nilpotently on the Lie algebra $\mathrm{Der}(V)$, so that $\exp(\mathrm{ad}\, D)$ is an automorphism of $\mathrm{Der}(V)$. Show that this automorphism is conjugation by $\exp(D)$.