SYLLABUS FOR ANNUAL FOUNDATION SCHOOLS

In each of the three schools, topics are typically bunched into 4 modules of 6 hours each (except in 1.3 Topology) so as to facilitate four different speakers for each modules. For less number of speakers, the organizers are free to distribute the material suitably. In 1.3 Topology, there are only three modules, and it is prefered if only three speakers share it. The speakers are supposed to coordinate with each other. Each speaker is requested to cover the material as far as possible with more emphasis on ideas, examples and exercises, than reproducing proofs from standard texts. Also, an earlier topic which could not be covered by an earlier speaker for some reason, but required currently, should also covered.

In AFS II and III, we presume that the participants have attended the previous AFS’s. However, speakers should be sympathetic to those who could not attend the previous AFS’s and try to help them out with the relevant/necessary material.

1. Annual Foundation School-I

1.1. Group Theory.

(1) **Basic examples** of groups such as cyclic groups, dihedral and quaternion groups, matrix groups and permutation groups. review of normal subgroups and isomorphism theorems, internal and direct products.

(2) **Isometries** of $\mathbb{R}^n$ and plane. group actions, finite subgroups of $SO(2)$ and $SO(3)$.

(3) **Sylow Theory**, classification of finite groups of order 12, simplicity of the alternating groups and PSL(V) solvable groups, p-groups, Jordan-Hölder theorem.

(4) **Linear groups** Classical groups, $SU(2)$, latitudes and longitudes on the 3-sphere, simplicity of $SO(3)$, normal subgroups of $SL(2, F)$.

References:


1.2. **Real Analysis**.

1. **Basics** Abstract measure spaces and the concept of measurability, simple functions, basic properties of measures, Lebesgue integration of positive functions and complex values functions, measure zero sets, completion of a measure and outer measure.

2. **Positive Borel Measures**: Topological preliminaries on locally compact Hausdorff spaces, Riesz representation theorem (outline of the proof), Borel measures, Lebesgue measure on $\mathbb{R}^k$, comparison with Riemann integration. Approximation by continuous functions, Generalized Riesz representation theorem.


4. **Integration on products** Monotone classes, algebra on products, product measure, Fibini, completion, convolution.

**References:**


1.3. **Topology**.

1. **Review of Multivariable Differential Calculus (8 hours)** Differentiability of functions on open subsets of $\mathbb{R}^n$, relation with partial/directional derivative, Taylor’s theorem etc.

   - Inverse and implicit function theorems, rank theorem,
   - Differentiability of functions on arbitrary subsets of subsets of $\mathbb{R}^n$, diffeomorphisms, smooth version of invariance of domain.

   Richness of smooth functions, smooth partition of unity and consequences on subspaces of $\mathbb{R}^n$ such as approximation of continuous functions by smooth functions.

   - Sard’s theorem for smooth functions $\mathbb{R}^n \to \mathbb{R}^m$ and some applications.

2. **Basic point-set topology part (a) (8 hours)**

   - Open sets and closed sets, limit points, closure and boundary points, subspace. Bases and subbases.

   Continuous functions, open functions, closed functions, homeomorphisms.
Separation axioms: Hausdorffness regularity and normality. Urysohn’s lemma and Tietze extension theorem.
Compactness and Lindelöf property, local compactness.
I and II countability separability.
Path connectedness, connectedness, local connectedness.
Product topology.

(3) **Basic point-set topology part (b) (8 hours)**
Induced and coinduced topologies.
Quotient topology, separation axioms under quotient topology, criterion for a restriction of a quotient map to be a quotient map, examples such as cones, cylinders, Mobius strips etc.
Paracompactness and partition of unity, Stone’s theorem (paracompactness of metric topology).
Topological groups and orbit spaces. Examples from matrix groups.
Function spaces, compact-open-topology and exponential correspondence.

**References:**


2. Annual Foundation School II

2.1. **Ring Theory.**

(1) **Modules over Principal Ideal Domains** Modules, direct sums, free modules, finitely generated modules over a PID, structure of finitely generated abelian groups, rational and Jordan canonical form.
2.2. Functional Analysis.

(1) **Normed linear spaces**, Continuous linear transformations, application to differential equations, Hahn-Banach theorems-analytic and geometric versions, vector valued integration.

(2) **Bounded Linear maps on Banach Spaces** Baire’s theorem and applications: Uniform boundedness principle and application to Fourier series, Open mapping and closed graph theorems, annihilators, complemented subspaces, unbounded operators and adjoints.

(3) **Bounded linear functionals** Weak and weak* topologies, Applications to reflexive separable spaces, Uniformly convex spaces, Application to calculus of variations.

(4) **Hilbert spaces**, Riesz representation theorem, Lax-Milgram lemma and application to variational inequalities, Orthonormal bases, Applications to Fourier series and examples of special functions like Legendre and Hermite polynomials.

References


2.3. Differential Topology.

(1) Review of differential calculus of several variables: Inverse and implicit function theorems. Richness of smooth functions; smooth partition of unity, Submanifolds of Euclidean spaces (without and with boundary) Tangent space, embeddings, immersions and submersions, Regular values, pre-image theorem, Transversality and Stability. [The above material should be supported amply by exercises and examples from matrix groups.]


(4) Intersection Theory: Transverse homotopy theorem and oriented intersection number. Degree of maps both oriented and non oriented cases, winding number, Jordan Brouwer separation theorem, Borsuk-Ulam theorem.

References:


3. Annual Foundation School-III

3.1. Field Theory. Module 1:

(1) Field extension and examples. Algebraic and transcendental elements, minimal polynomial. Degree of a field extension, finite and infinite extensions. Simple extensions.

(2) Transitivity of finite/algebraic extensions. Compositum of two fields.

(3) Ruler and compass constructions. Characterization of constructible numbers via square root towers of fields. Impossibility of squaring the circle, trisection of angles and duplication of cubes by ruler and compass.
(4) Gauss’ criterion of constructible regular polygons. (Wantzel’s characterization of constructible regular \( p \)-polygons. Richmond’s construction of a regular pentagon.)

(5) Examples of symmetric polynomials. The fundamental theorem of symmetric polynomials. Newton’s identities for power sum symmetric polynomials.


Module 2:

(7) Splitting fields and algebraic closures; existence and isomorphisms.


(9) Separable and inseparable extensions. Transitivity of separable extensions. Roots of an irreducible polynomial have equal multiplicity. Separable degree.

(10) Finite fields: existence and uniqueness, algebraic closure. Finite subgroup of the multiplicative group of a field is cyclic. Gauss’ formula for the number of monic irreducible polynomials of a given degree over a finite field.

(11) Factorization of polynomials over finite fields. Primitive element theorem. Finite separable extensions have a primitive element.

(12) Normal extensions and their examples. Characterization of normal extensions in terms of embeddings and splitting fields.

Module 3:


(14) Artin’s Theorem about fixed field of a finite group of automorphisms. Behavior of Galois group under isomorphisms. Normal subgroups of the Galois groups and their fixed fields.


(16) Roots of unity in a field. Galois group of \( x^n - a \) over a field having \( n^{th} \) roots of unity. Irreducibility of the cyclotomic polynomial \( \Phi_n(x) \) over \( \mathbb{Q} \). A recursive formula for \( \Phi_n(x) \).

(17) Discriminant of \( \Phi_p(x) \). Subfields of \( \mathbb{Q}(\zeta_p) \). Kronecker-Weber Theorem
for quadratic extensions of \( \mathbb{Q} \). Algorithm for construction of primitive elements of subfields of \( \mathbb{Q}(\zeta_p) \). Subfields of \( \mathbb{Q}(\zeta_7) \), \( \mathbb{Q}(\zeta_{13}) \) and \( \mathbb{Q}(\zeta_{17}) \).

(18) Infinitude of primes \( p \equiv 1 \pmod{n} \). Inverse Galois problem for finite abelian groups. Structure of some cyclic extensions.

**Module 4:**

(19) Cyclic extensions of degree \( p \) over fields with characteristics \( p \). Solvable groups. Simplicity of \( A_n \).

(20) Galois group of composite extensions Galois closure of a separable field extension. (21) Radical extensions. Solvability by radicals and solvable Galois groups. A quintic polynomial which is not solvable by radicals.


(23) Galois group as a group of permutations. Irreducibility and transitivity. Galois groups of quartics.

(24) The norm and the trace function. Multiplicative form of Hilbert’s Theorem 90. Cyclic extensions of degree \( n \). Additive version of Hilbert’s 90. Cyclic extensions of prime degree: Artin-Schreier Theorem.

**References:**


3.2. **Complex Analysis.**

1. **Quick review of algebra and topology of complex plane,** sequences and series, uniform convergence, Weierstrass M-test.
   Complex differentiability, basic properties, analytic functions: power series, Abel’s theorem, examples, Cauchy-Riemann equations: Cauchy derivative versus Frechet derivative.
   Geometric interpretation of holomorphy, formal differentiation, Mobius transformation and the Riemann sphere.

2. **Line integrals,** basic properties, differentiation under integral sign.
   Isolated singularities, Laurent series and residues.
   Winding number and argument principle.

3. **Homotopy and Homology versions** of Cauchy’s theorem, Inverse function theorem, Rouche’s theorem. Schwarz’s lemma.
(4) **Harmonic functions:** Mean Value property maximum principle etc. Schwarz reflection principle. Harnack’s Principle, Subharmonic functions, Dirichlet’s problem, Perron’s solution, Green’s function and an outline of proof of Riemann Mapping theorem.

**References**

2. T. W. Gamelin, Complex Analysis Springer-Verlag, UTM.

3.3. **Algebraic Topology.**

(1) **Statement of basic problems in Algebraic Topology:** extension problems and lifting problems; homotopy, relative homotopy, deformation, contraction, retracts etc.

Typical constructions: Adjunction spaces, Mapping cones, Mapping cylinder, Smash-product, reduced cones reduced suspension etc.

Categories and Functors. Definition and examples. Equivalence of functors, adjoint functors, examples.

Computation of fundamental group of the circle and applications.


Applications: cellular Approximation theorem, Brouwer’s invariance of domain etc.

(3) **Covering Spaces and Fundamental group**, lifting properties, relation with fundamental group.

Classification of covering spaces (proof of existence may be skipped),

Computation of Fundamental groups: simpler cases of Van-Kampen theorem. Effect of attaching $n$-cells, (4.6).

(4) **Singular and Simplicial Homology** Chain complexes, exact sequences of complexes, snake lemma, four lemma and five lemma, homology long exact sequence.

Axioms for homology, construction of singular chain complex, verification of axioms (except homotopy axiom and excision axiom).

Simplicial and singular simplicial homologies. Statement of equivalence of all these homologies.
Computations and applications: Separation theorems, Invariance of Domain. Euler characteristic.

References: