SYLLABUS FOR ANNUAL FOUNDATION SCHOOLS

In each of the three schools, topics are typically bunched into 4 modules of **6 hours** each (except in 1.3 Topology) so as to facilitate four different speakers for each modules. For less number of speakers, the organizers are free to distribute the material suitably. In 1.3 Topology, there are only three modules, and it is prefered if only three speakers share it. The speakers are supposed to coordinate with each other. Each speaker is requested to cover the material as far as possible with more emphasis on *ideas, examples and exercises,* than reproducing proofs from standard texts. Also, an earlier topic which could not be covered by an earlier speaker for some reason, but required currently, should also covered.

In AFS II and III, we presume that the participants have attended the previous AFS's. However, speakers should be sympathetic to those who could not attend the previous AFS's and try to help them out with the relevant/necessary material.

1. Annual Foundation School-I

1.1. Group Theory.

(1) **Basic examples** of groups such as cyclic groups, dihedral and quaternion groups, matrix groups and permutation groups. review of normal subgroups and isomorphism theorems, internal and direct products.
(2) **Isometries** of $\mathbb{R}^n$ and plane. group actions, finite subgroups of $SO(2)$ and $SO(3)$.
(3) **Sylow Theory,** classification of finite groups of order 12, simplicity of the alternating groups and PSL(V) solvable groups, p-groups, Jordan-Hölder theorem.
(4) **Linear groups** Classical groups, $SU(2)$, latitudes and longitudes on the 3-sphere, simplicity of $SO(3)$, normal subgroups of $SL(2, F)$.

References:

(1) M. Artin, Algebra, Second Edition, Prentice Hall of India, 2011.
(2) N. Jacobson, Basic Algebra Volume 1, Second Edition, Dover Publications, 2009.
(3) S. Lang, Algebra, Third Edition, Springer India, 2001.

1.2. **Real Analysis.**

(1) **Basics** Abstract measure spaces and the concept of measurability, simple functions, basic properties of measures, Lebesgue integration of positive functions and complex values functions, measure zero sets, completion of a measure and outer measure.
(2) **Positive Borel Measures:** Topological preliminaries on locally compact Hausdorff spaces, Riesz representation theorem (outline of the proof), Borel measures, Lebesgue measure on $\mathbb{R}^k$, comparison with Riemann integration. Approximation by continuous functions, Generalized Riesz representation theorem.
(3) **Differentiation** Maximal functions, Lebesgue points, I Fundamental Theorem of integral calculus, Absolutely continuous functions, II fundamental theorem of integral calculus. Change of variable formula.
(4) **Integration on products** Monotone classes, algebra on products, product measure, Fibini, completion, convolution.

References:

(1) I. K. Rana, *Introduction to Measure and Integration,* II edition, Narosa Publishing House, New Delhi.

(2) Royden H. L. *Real Analysis*, III edition, Macmillan, New York, 1963.
(3) W. Rudin, *Real And Complex Analysis*, III edition, Tata McGraw-Hill Higher Education, New Delhi 1987.

1.3. **Topology.**

(1) **Review of Multivariable Differential Calculus (8 hours)** Differentiability of functions on open subsets of $\mathbb{R}^n$, relation with partial/directional derivative, Taylor's theorem etc.

    Inverse and implicit function theorems, rank theorem,

    Differentiability of functions on arbitrary subsets of subsets of $\mathbb{R}^n$, diffeomorphisms, smooth version of invariance of domain.

    Richness of smooth functions, smooth partition of unity and consequences on subspaces of $\mathbb{R}^n$ such as approximation of continuous functions by smooth functions.

    Sard's theorem for smooth functions $\mathbb{R}^n \to \mathbb{R}^m$ and some applications.
(2) **Basic point-set topology part (a) (8 hours)**

    Open sets and closed sets, limit points, closure and boundary points, subspace. Bases and subbases.

    Continuous functions, open functions, closed functions, homeomorphisms.

Separation axioms: Hausdorffness regularity and normality. Urysohn's lemma and Tietze extension theorem.

Compactness and Lindelöff property, local compactness.

I and II countability separability.

Path connectedness, connectedness, local connectedness.

Product topology.

(3) **Basic point-set topology part (b) (8 hours)**

Induced and coinduced topologies.

Quotient topology, separation axioms under quotient topology, criterion for a restriction of a quotient map to be a quotient map, examples such as cones, cylinders, Mobius strips etc.

Paracompactness and partition of unity, Stone's theorem (paracompactness of metric topology).

Topological groups and orbit spaces. Examples from matrix groups.

Function spaces, compact-open-topology and exponential correspondence.

References:

(1) M.A. Armstrong, *Basic Topology*, Springer.
(2) K. D. Joshi, *Introduction to General Topology*, New Age International (P) Limited.
(3) J. R. Munkres, *Topology, A First Course*, Prentice-Hall of India, New Delhi 1987.
(4) W. Rudin, *Principles of Mathematical Analysis* 3rd edition, McGraw Hill, 1976.
(5) A. R. Shastri, *Elements of Differential Topology,* CRC Press, Taylor and Francis Group, Boca Raton, 2011.
(6) A. R. Shastri, *Basic Algebraic Topology,* CRC Press, Taylor and Francis Group, Boca Raton. 2013.
(7) M. Spivak, *Calculus on Manifolds*, Benjamin Ink., New York, 1965.

2. **Annual Foundation School II**

2.1. **Ring Theory.**

(1) **Modules over Principal Ideal Domains** Modules, direct sums, free modules, finitely generated modules over a PID, structure of finitely generated abelian groups, rational and Jordan canonical form.

(2) **Basics** Commutative rings, nil radical, Jacobson radical, localization of rings and modules, Noetherian rings, primary decomposition of ideals and modules.
(3) **Integral extensions** of rings, Going up and going down theorems, finiteness of integral closure, discrete valuation rings, Krull's normality criterion, Noether normalization lemma, Hilbert's Nullstellensatz
(4) **Semisimple rings,** Wedderburn's Theorem, Rings with chain conditions and Artin's theorem, Wedderburn's main theorem,

References:

(1) S. Lang, Algebra, 3rd edition, Addison-Wesley.
(2) D. S. Dummit and R. M. Foote, Abstract Algebra, 2nd edition John-Wiley.
(3) N. Jacobson, Basic Algebra, Vol. 1 and 2, Dover, 2011.
(4) A. W. Knapp, Advanced Algebra, Birkhauser, 2011.

2.2. **Functional Analysis.**

(1) **Normed linear spaces,** Continuous linear transformations, application to differential equations, Hahn-Banach theorems-analytic and geometric versions, vector valued integration.
(2) **Bounded Linear maps on Banach Spaces** Baire's theorem and applications: Uniform boundedness principle and application to Fourier series, Open mapping and closed graph theorems, annihilators, complemented subspaces, unbounded operators and adjoints
(3) **Bounded linear functionals** Weak and weak* topologies, Applications to reflexive separable spaces, Uniformly convex spaces, Application to calculus of variations
(4) **Hilbert spaces,** Riesz representation theorem, Lax-Milgram lemma and application to variational inequalities, Orthonormal bases, Applications to Fourier series and examples of special functions like Legendre and Hermite polynomials.

References

(1) J. B. Conway, *A Course in Functional Analysis,* II edition, Springer, Berlin 1990.
(2) C. Goffman, G. Pedrick, *First Course in Functional Analysis,* Prentice-Hall, 1974.
(3) S. Kesavan, *Functional Analysis* Volume 52 of Texts and readings in mathematics, Hindustan Book Agency (India), 2009.
(4) B. B. Limaye, *Functional Analysis,* II edition New Age International,1996.

(5) A. Tayor and D. Lay, *Introduction to Functional Analysis*, Wiley, New York, 1980.

2.3. **Differential Topology.**

(1) **Review of differential calculus of several variables**: Inverse and implicit function theorems. Richness of smooth functions; smooth partition of unity, Submanifolds of Euclidean spaces (without and with boundary) Tangent space, embeddings, immersions and submersions, Regular values, pre-image theorem, Transversality and Stability. [The above material should be supported amply by exercises and examples from matrix groups.]
(2) **Abstract topological and smooth manifolds**, partition of unity, Fundamental gluing lemma with criterion for Hausdorffness of the quotient, classification of 1-manifolds. Definition of a vector bundle and tangent bundle as an example. Sard's theorem. Easy Whitney embedding theorems.
(3) **Vector fields and isotopies** Normal bundle and Tubular neighbourhood theorem. Orientation on manifolds and on normal bundles. Vector fields. Isotopy extension theorem. Disc Theorem. Collar neighbourhood theorem.
(4) **Intersection Theory:** Transverse homotopy theorem and oriented intersection number. Degree of maps both oriented and non oriented cases, winding number, Jordan Brouwer separation theorem, Borsuk-Ulam theorem.

References:

(1) V. Gullemin and A. Pollack, Differential Topology, Englewood Cliff, N.J. Prentice Hall (1974).
(2) W. Hirsch, Differential Topology, Springer-Verlag.
(3) J. W. Milnor, Topology from the Differential Viewpoint, Univ. Press, Verginia.
(4) Anant R. Shastri, Elements of Differential Topology, CRC Press, 2011.

3. **Annual Foundation School-III**

3.1. **Field Theory. Module 1:**
(1) Field extension and examples. Algebraic and transcendental elements, minimal polynomial. Degree of a field extension, finite and infinite extensions. Simple extensions.
(2) Transitivity of finite/algebraic extensions. Compositum of two fields.
(3) Ruler and compass constructions. Characterization of constructible numbers via square root towers of fields. Impossibility of squaring the circle, trisection of angles and duplication of cubes by ruler and compass.

(4) Gauss' criterion of constructible regular polygons. (Wantzel's characterization of constructible regular p-polygons. Richmond's construction of a regular pentagon.)
(5) Examples of symmetric polynomials. The fundamental theorem of symmetric polynomials. Newton's identities for power sum symmetric polynomials.
(6) Discriminant in terms of power-sum symmetric polynomials. Discriminant of a cubic. Existence of a splitting field of a polynomial. Fundamental theorem of algebra via symmetric polynomials.

Module 2:
(7) Splitting fields and algebraic closures; existence and isomorphisms.
(8) Criterion for multiple roots of polynomials in terms of derivatives. Characterization of perfect fields of positive characteristic.
(9) Separable and inseparable extensions. Transitivity of separable extensions. Roots of an irreducible polynomial have equal multiplicity. Separable degree.
(10) Finite fields: existence and uniqueness, algebraic closure. Finite subgroup of the multiplicative group of a field is cyclic. Gauss' formula for the number of monic irreducible polynomials of a given degree over a finite field.
(11) Factorization of polynomials over finite fields. Primitive element theorem. Finite separable extensions have a primitive element.
(12) Normal extensions and their examples. Characterization of normal extensions in terms of embeddings and splitting fields.

Module 3:
(13) Galois extensions. Galois groups of finite extensions of finite fields and quadratic extensions. Galois groups of biquadratic extensions. Galois groups of a separable cubic polynomials. Fundamental Theorem of Galois theory (FTGT).
(14) Artin's Theorem about fixed field of a finite group of automorphisms. Behavior of Galois group under isomorphisms. Normal subgroups of the Galois groups and their fixed fields.
(15) Fundamental theorem of algebra via FTGT.
Gauss' criterion for constructible regular polygons. Symmetric rational functions. Galois group of some binomials.
(16) Roots of unity in a field. Galois group of $x^n - a$ over a field having n^{th} roots of unity. Irreducibility of the cyclotomic polynomial $\Phi_n(x)$ over $\mathbb{Q}$. A recursive formula for $\Phi_n(x)$.
(17) Discriminant of $\Phi_p(x)$. Subfields of $\mathbb{Q}(\zeta_p)$. Kronecker-Weber Theorem

for quadratic extensions of $\mathbb{Q}$. Algorithm for construction of primitive elements of subfields of $\mathbb{Q}(\zeta_p)$. Subfields of $\mathbb{Q}(\zeta_7)$, $\mathbb{Q}(\zeta_{13})$ and $\mathbb{Q}(\zeta_{17})$.
(18) Infinitude of primes $p \equiv 1 \pmod{n}$. Inverse Galois problem for finite abelian groups. Structure of some cyclic extensions.

Module 4:
(19) Cyclic extensions of degree p over fields with characteristics p. Solvable groups. Simplicity of A_n.
(20) Galois group of composite extensions Galois closure of a separable field extension. (21) Radical extensions. Solvability by radicals and solvable Galois groups. A quintic polynomial which is not solvable by radicals.
(22) Cardano's method for roots of cubic equations. Lagrange's method for roots of quartic equations. Ferrari's method for roots of quartic equations.
(23) Galois group as a group of permutations. Irreducibility and transitivity. Galois groups of quartics.
(24) The norm and the trace function. Multiplicative form of Hilbert's Theorem 90. Cyclic extensions of degree n. Additive version of Hilbert's 90. Cyclic extensions of prime degree: Artin-Schreier Theorem.
References:

(1) M. Artin, Algebra, II Edition, Prentice Hall of India, 2011.
(2) N. Jacobson, Basic Algebra I, II Edition, Dover Publications, 2009.
(3) S. Lang, Algebra, III Edition, Springer, 2002.

3.2. **Complex Analysis.**

(1) **Quick review of algebra and topology of complex plane**, sequences and series, uniform convergence, Weierstrass M-test.
Complex differentiability, basic properties, analytic functions: power series, Abel's theorem, examples, Cauchy-Riemann equations: Cauchy derivative versus Frechet derivative.
Geometric interpretation of holomorphy, formal differentiation, Mobius transformation and the Riemann sphere.
(2) **Line integrals,** basic properties, differentiation under integral sign.
Primitive existence theorem, Cauchy-Goursat theorem (statements and sketch of the proof only) Cauchy's theorem on a convex domain theorem. Cauchy's integral formula, Taylor's theorem, Liouville, Maximum modulus principle. Zeros of Holomorphic functions, identity theorem, open mapping theorem.
Isolated singularities, Laurent series and residues.
Winding number and argument principle.
(3) **Homotopy and Homology versions** of Cauchy's theorem, Inverse function theorem, Rouche's theorem. Schwarz's lemma.

(4) **Harmonic functions:** Mean Value property maximum principle etc. Schwarz reflection principle. Harnack's Principle, Subharmonic functions, Dirichlet's problem, Perron's solution, Green's function and an outline of proof of Riemann Mapping theorem.

References

(1) 1. J. B. Conway, Functions of One complex Variable, II edition GTM 11 Springer-Verlag (1973).
(2) 2. T. W. Gamelin, Complex Analysis Springer-Verlag, UTM.
(3) 3. R. Remmert, Theory of Complex Functions, GTM-122 Springer-Verlag (1980)
(4) 4. Anant R. Shastri, Basic Complex Analysis of One Variable, MacMillan Publishers India Ltd, Delhi (2011)

3.3. **Algebraic Topology.**

(1) **Statement of basic problems in Algebraic Topology:** extension problems and lifting problems; homotopy, relative homotopy, deformation, contraction, retracts etc.

Typical constructions: Adjunction spaces, Mapping cones, Mapping cylinder, Smash-product, reduced cones reduced suspension etc.

Categories and Functors. Definition and examples. Equivalence of functors, adjoint functors, examples.

Computation of fundamental group of the circle and applications.

(2) **CW-complexes Simplicial complexes** basic topological properties of CW complexes. Products of CW complexes (especially the CW-structure on $X \times [0, 1]$). Homotopy theoretic properties of CW complexes. Abstract simplicial complexes and geometric realization, barycentric subdivision and simplicial approximation theorem.

Applications: cellular Approximation theorem, Brouwer's invariance of domain etc.

(3) **Covering Spaces and Fundamental group**, lifting properties, relation with fundamental group.

Classification of covering spaces(proof of existence may be skipped),

Computation of Fundamental groups: simpler cases of Van-Kampen theorem. Effect of attaching n-cells.(4.6).

(4) **Singular and Simplicial Homology** Chain complexes, exact sequences of complexes, snake lemma, four lemma and five lemma, homology long exact sequence.

Axioms for homology, construction of singular chain complex, verification of axioms (except homotopy axiom and excision axiom).

Simplicial and singular simplicial homologies. Statement of equivalence of all these homologies.

Computations and applications: Separation theorems, Invariance of Domain. Euler characteristic.

References:

(1) 1. A Hatcher, Algebraic Topology, Cambridge University Press.
(2) 2. Algebraic Topology, C. R. F. Maunder, Van Nostrand Reinhold Company. London.
(3) 3. E. H. Spanier, Algebraic Topology, Tata McGraw-Hill.
(4) 4. A. R. Shastri, Basic Algebraic Topology, CRC Press, Taylor and Francis Group 2013.