Set 5: Proof of Jordan - Brouwer separation theorem

Suppose $X \subseteq \mathbb{R}^{n+1}$ is a compact, connected n-manifold. Then $\mathbb{R}^{n+1} - X$ = disjoint union of two components
$= D_1 \amalg D_2$

One of the components (say D_1) is bounded and the other is unbounded. $\overline{D}_1$ is a manifold with boundary and $\partial \overline{D}_1 = X$.

1) Let $z \in \mathbb{R}^{n+1} - X$. Prove that for every $x \in X$, $\exists$ an open nbd. U of x in $\mathbb{R}^{n+1}$ such that there is a point of U that may be joined to z by a curve not intersecting X. (Hint: Consider the set of x such that this happens. Prove that it is open, closed and non-empty.)

2) Show that $\mathbb{R}^{n+1} - X$ has atmost 2 connected components. (Hint: take a small ball B such that $B - X$ has exactly two components. (1) $\Rightarrow$ z must be connected to one of the two components.)

3) If z_0, z_1 belongs to the same component of $\mathbb{R}^{n+1} - X$, then $W_2(X, z_0) = W_2(X, z_1)$.

(Hint: join z_0 to z_1 by a curve in $\mathbb{R}^{n+1} - X$

$$u_t = \frac{x - \gamma(t)}{|x - \gamma(t)|}$$ gives a homotopy.)

4) Let $z \in \mathbb{R}^{n+1} - X$. Let $v \in S^n$ and consider $r = \{ z + tv \mid t \geq 0 \}$. Prove that $r \pitchfork X$ iff v is a regular value of $u_z(x) = \dfrac{x - v}{|x - v|}$

5) Suppose r is a ray from z_0 as above intersecting X transversely. Let z_1 be a point on r not on X and suppose r intersects X at l points between z_0 and z_1. Prove that $W_2(f, z_0) = W_2(f, z_1) + l \pmod 2$

6) Prove that $\mathbb{R}^{n+1} - X$ has precisely two components $D_1 = \{ z \mid W_2(X, z) = 1 \}$ and $D_2 = \{ x \mid W_2(X, z) = 0 \}$

7) Prove that if z is large, $W_2(X, z) = 0$