Differential Topology Problems

Set 1

1. Exhibit a smooth map $\mathbb{R} \to \mathbb{R}$ such that the set of critical values is dense. [Hint: Enlist the rationals as $r_0, r_1, \dots$ and obtain a smooth map $[i, i+1] \to \mathbb{R}$ with critical value at r_i.]

2. Prove that S^n is simply connected if $n > 1$.
   [Hint: Apply Sard's theorem to a map $f: S^1 \to S^n$.]

3. Suppose $f_0, f_1 : X \to Y$ are homotopic. Prove that there exists a homotopy $F: X \times [0,1] \to Y$ such that $F(x,t) = f_0(x)$ if $t \leq 1/4$ and $F(x,t) = f_1(t)$ if $t \geq 3/4$.

4. A manifold is called contractible if the identity map is homotopic to a constant map. Prove that X is contractible $\iff$ $\forall\, Y$ any two maps $Y \to X$ is homotopic.

5. Prove that $\mathbb{R}^k$ is contractible.

Set 2

1. Prove that the antipodal map $x \mapsto -x$ of S^k is homotopic to the identity if k is odd.

2. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost all a in $\mathbb{R}^n$, $X + a$ intersects Y transversely.

3. Find which of the planes $\{ax + by + cz = d\}$ (for different a, b, c, d) intersects $S^2 = \{x^2 + y^2 + z^2 = 1\}$ transversely.

4. In the manifold $M \times M$, consider the submanifolds $M_c = M \times \{c\} \subset M \times M$ and $\Delta = \{(x, x) \in M \times M\}$. Prove that M_c is transverse to Δ.

5. Suppose $N \subset M$ of codimension $\geqslant 3$. Prove that $\pi_1(M)$ is isomorphic to $\pi_1(M - N)$.