The Belt Trick

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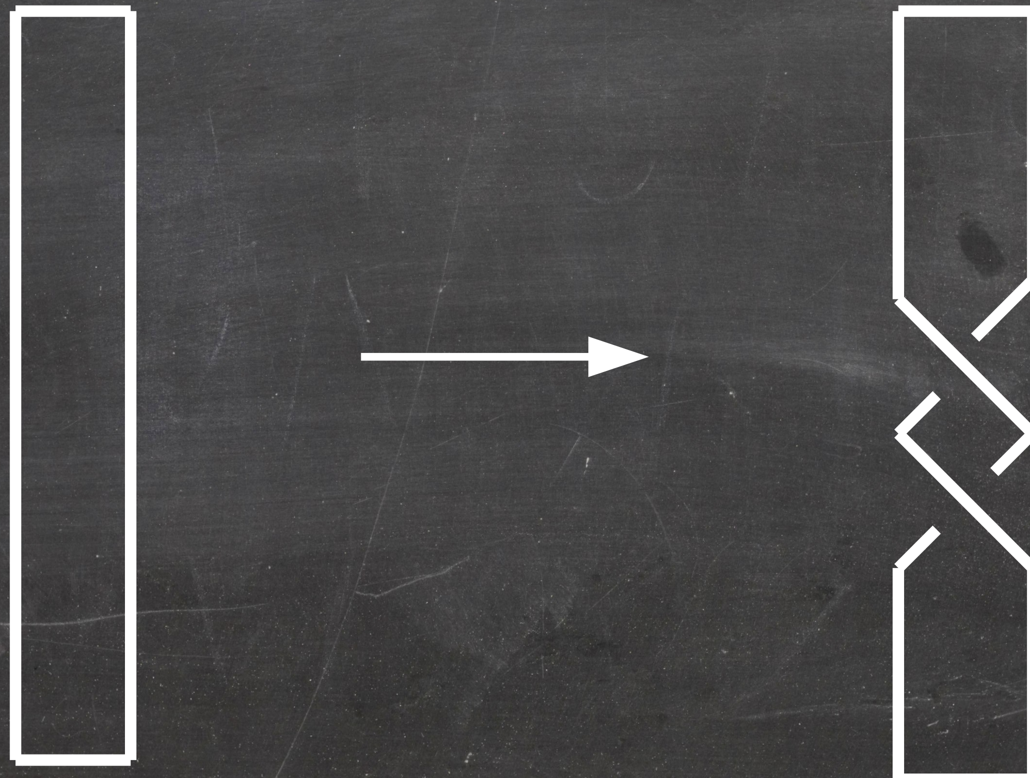

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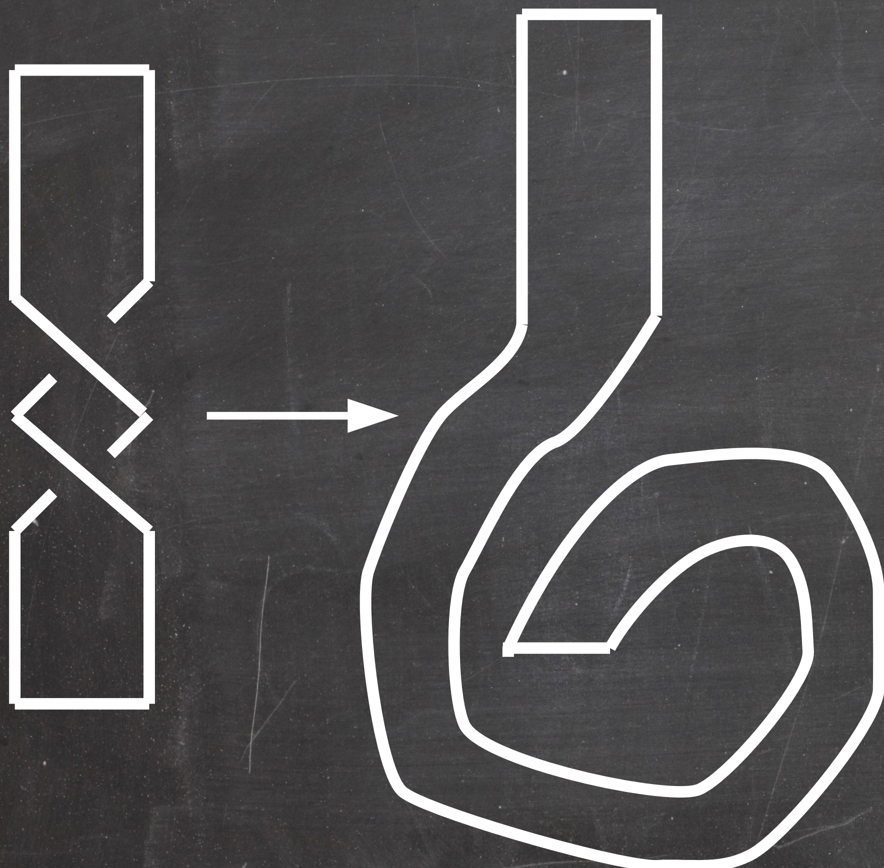

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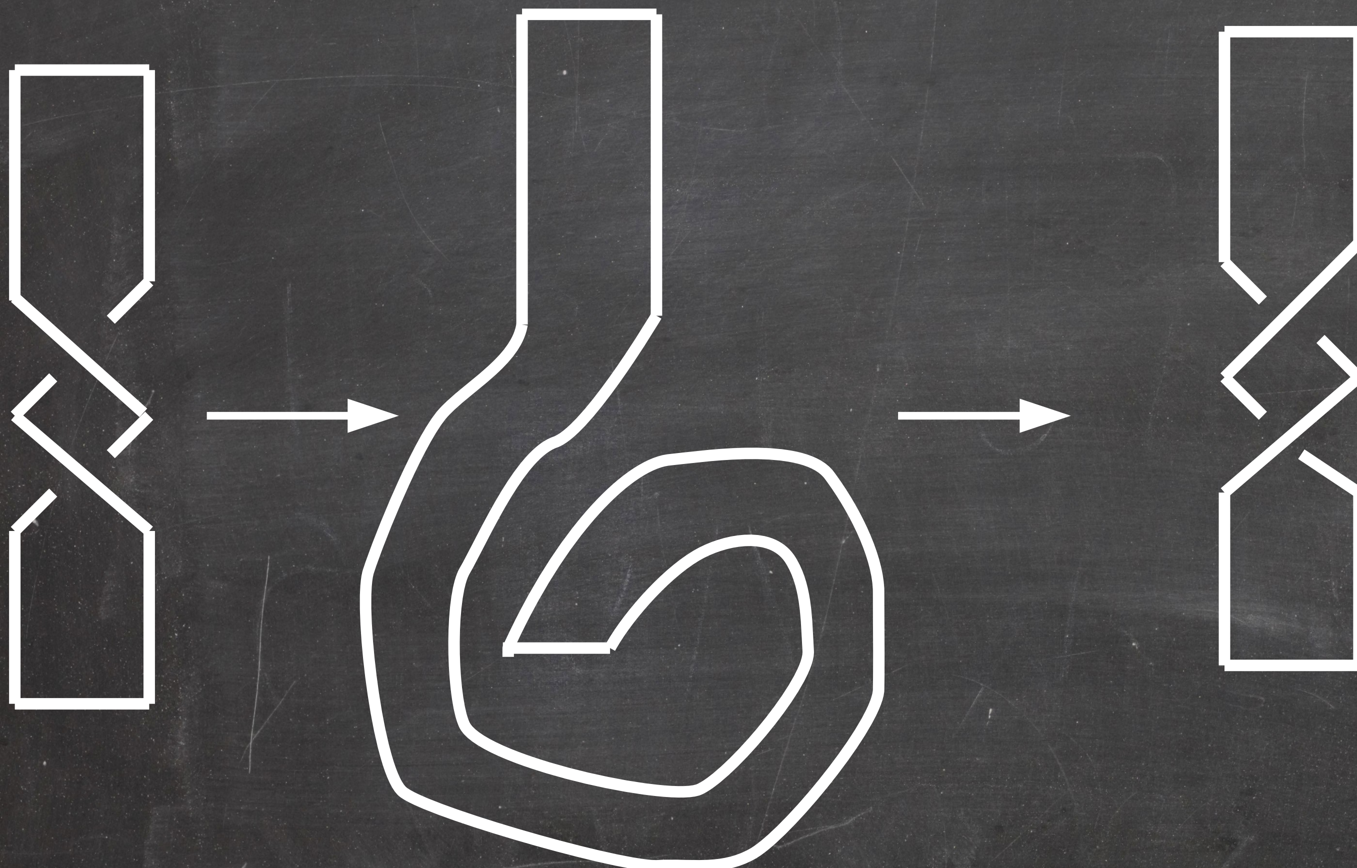

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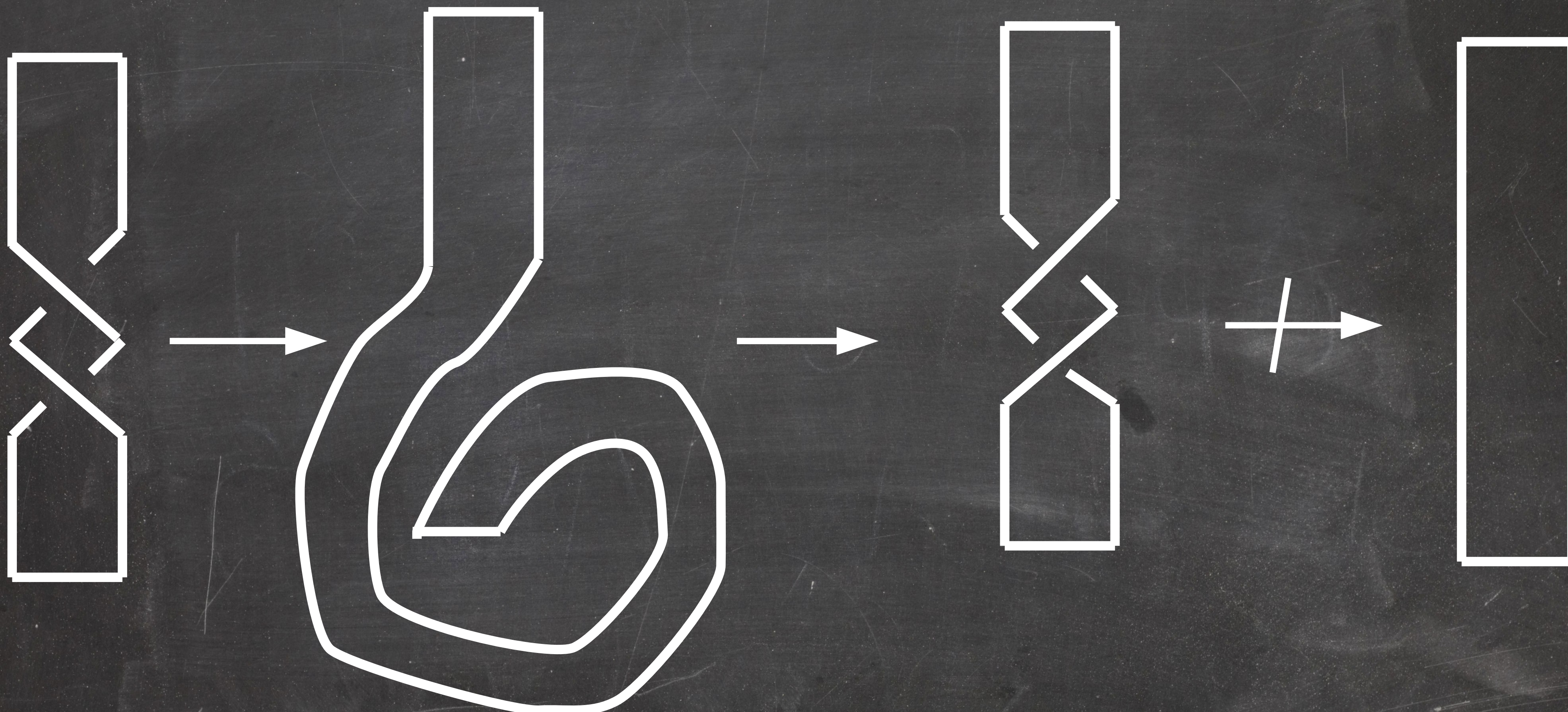

Observe that the twisted belt cannot be untwisted by these transformations

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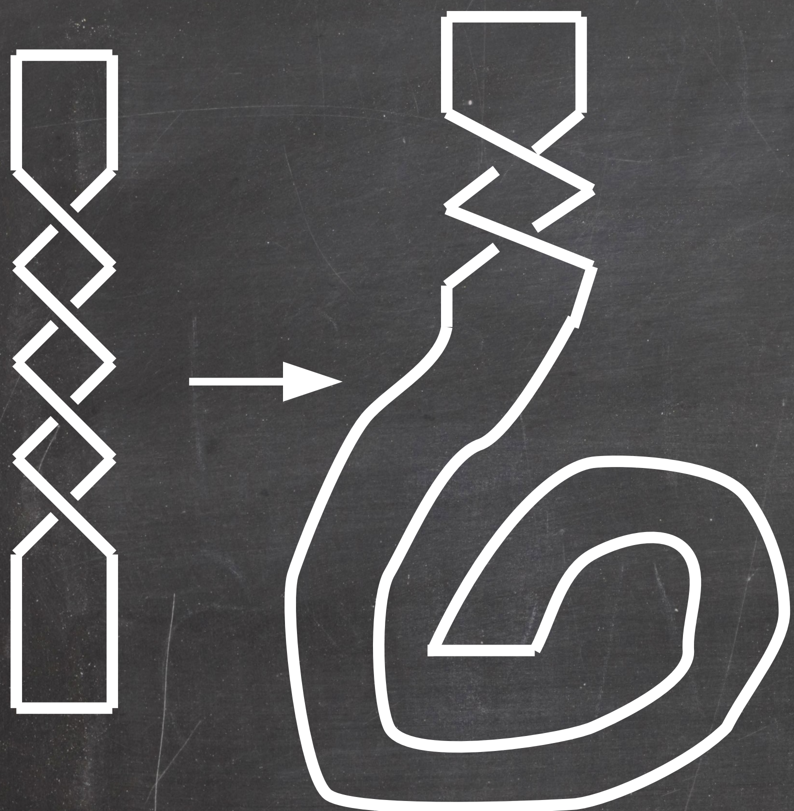

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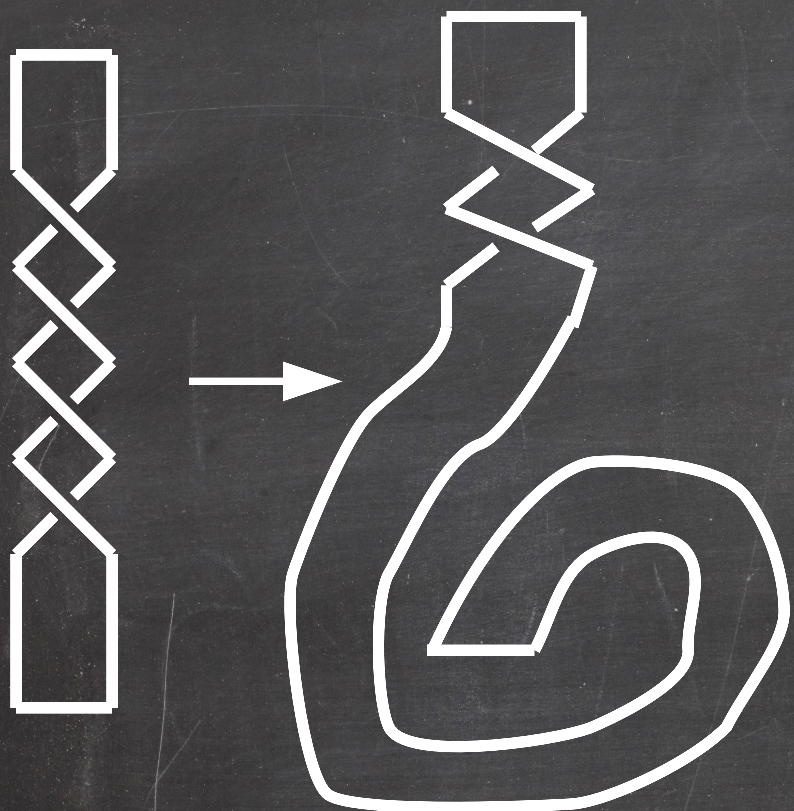

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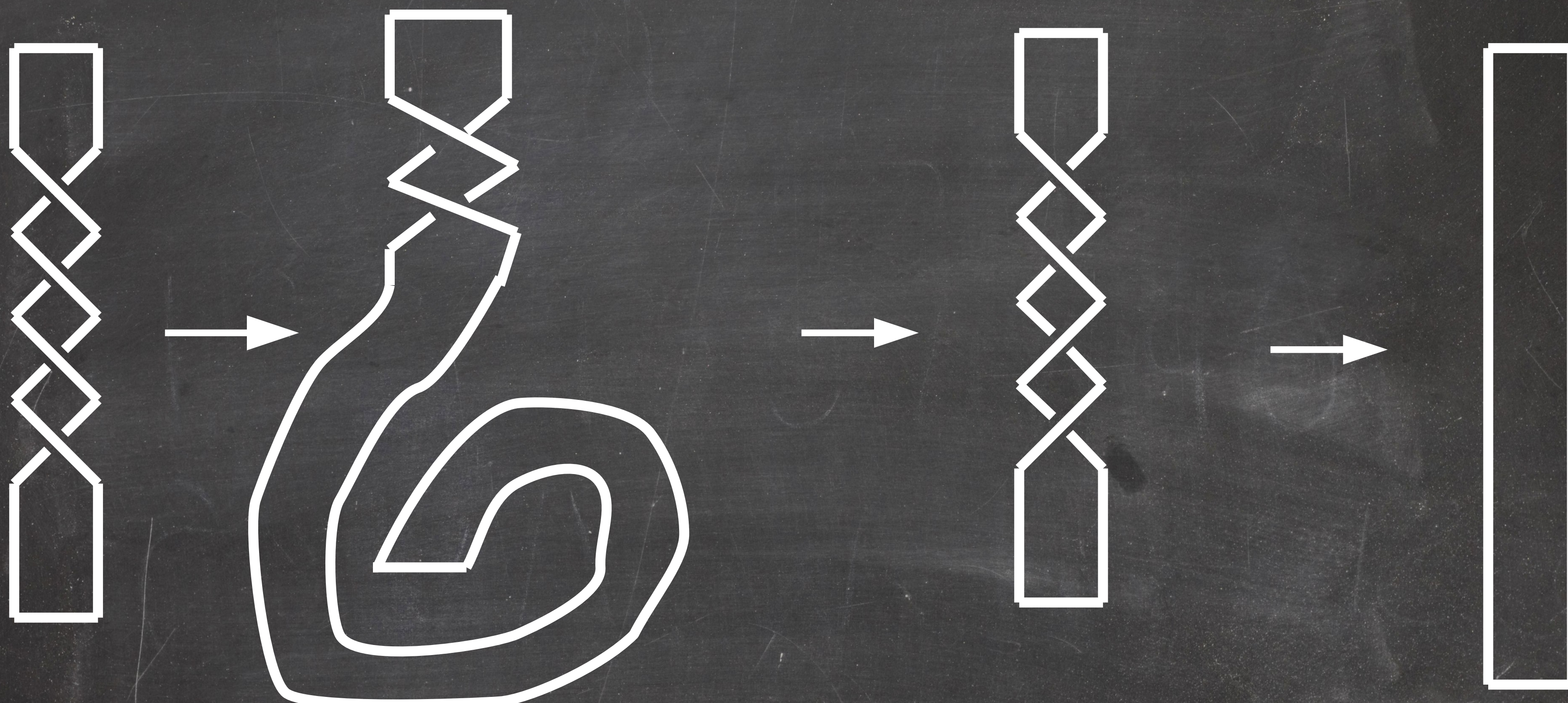

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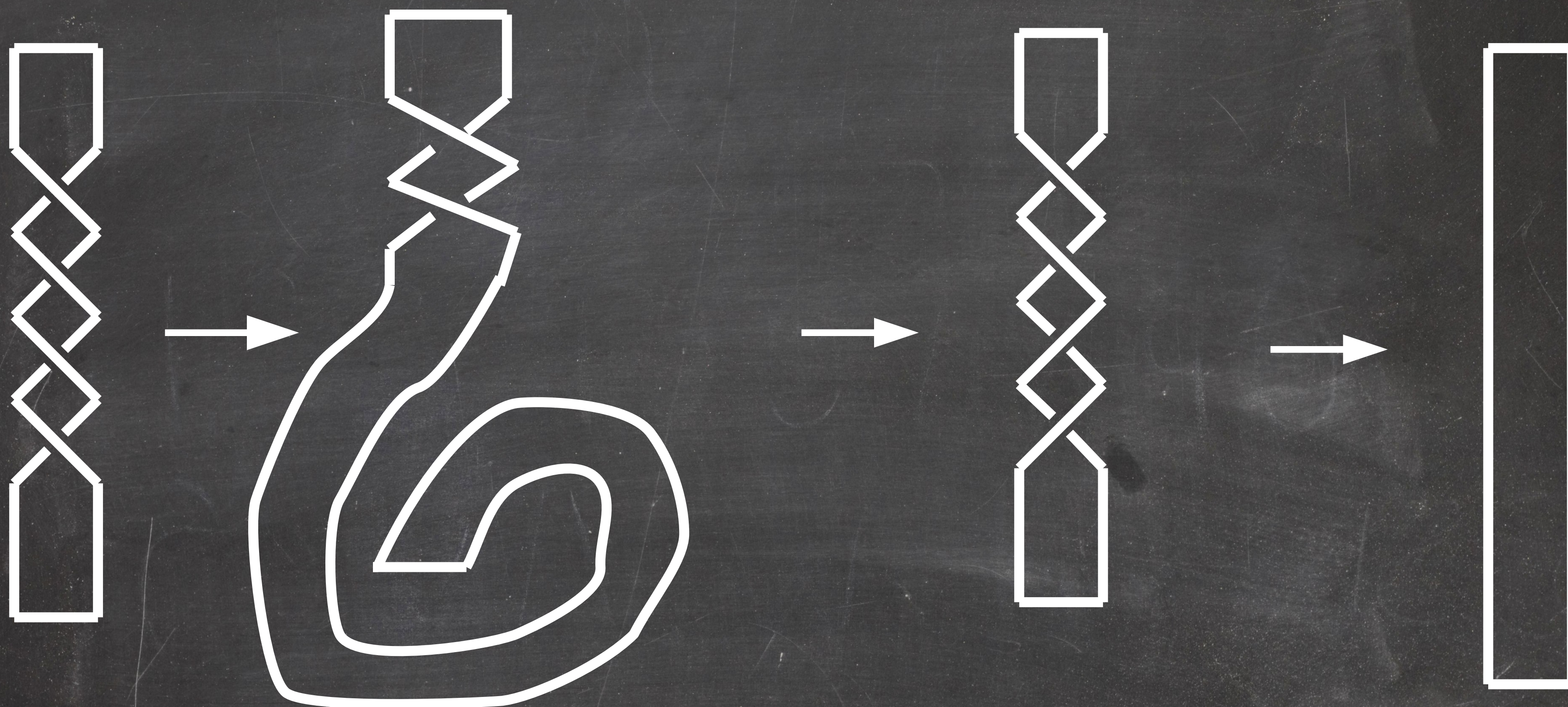
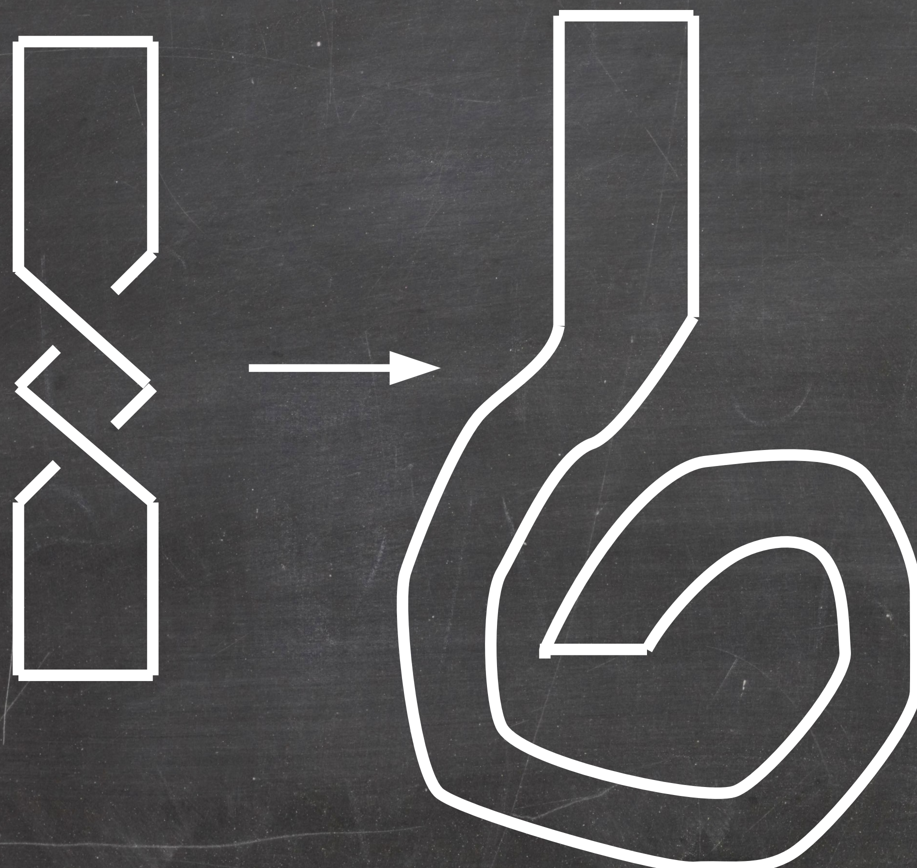


Hence the twice rotated belt can be transformed to the untwisted belt while the once rotated belt cannot. This is called THE BELT TRICK

In order to understand the Belt trick one needs to consider the transformations.



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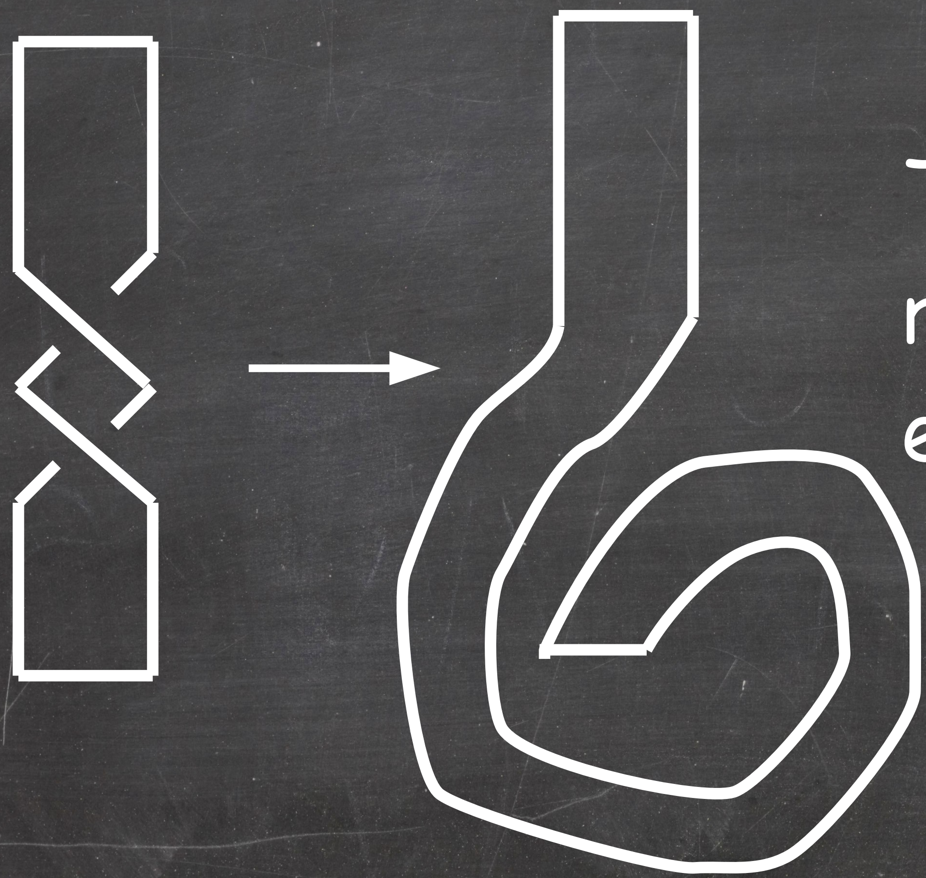

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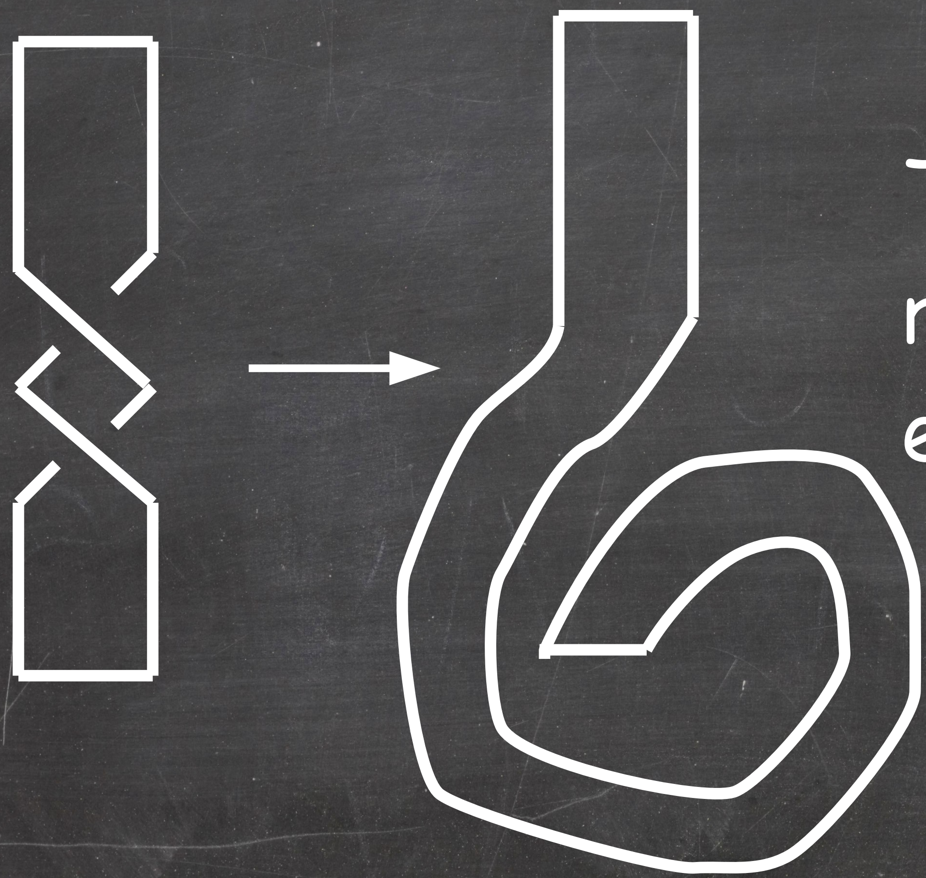

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One needs to construct an appropriate invariant.

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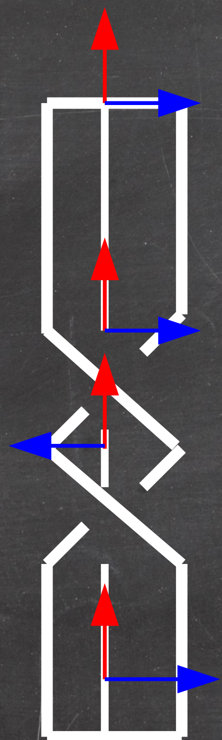

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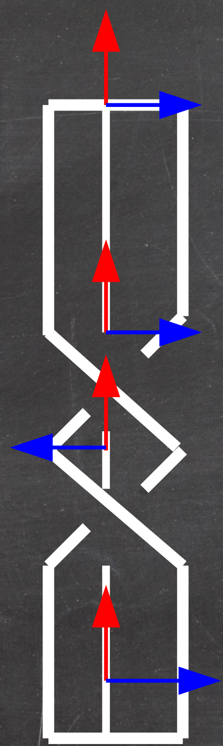

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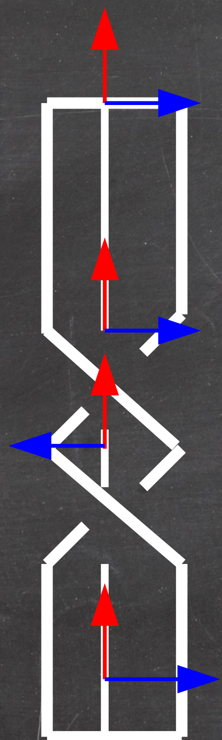

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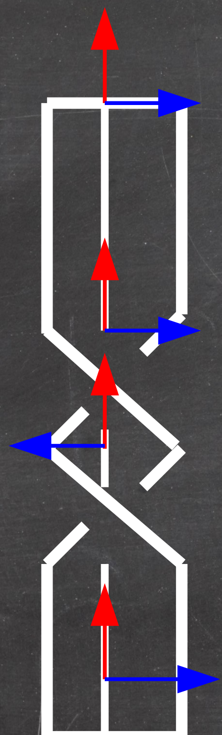

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Belt configurations ⟶ Maps from I to pairs of perpendicular vectors in R^3 with same value at 0 and 1.

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= S^2 (the sphere)

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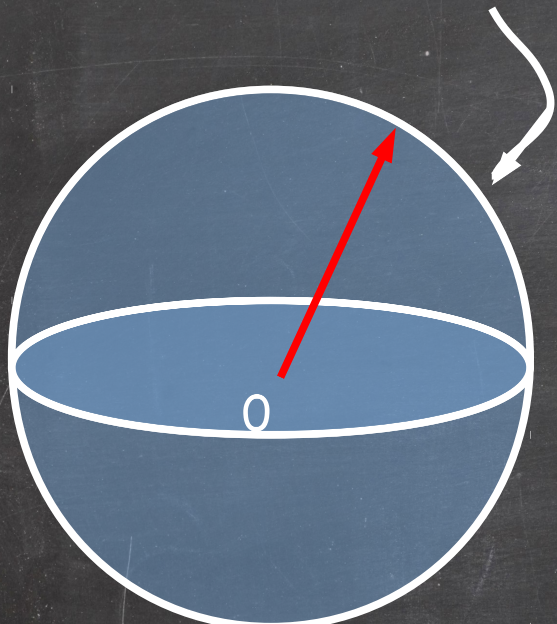

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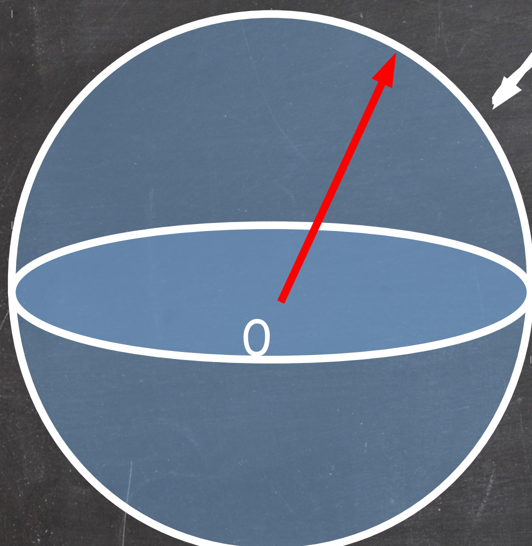

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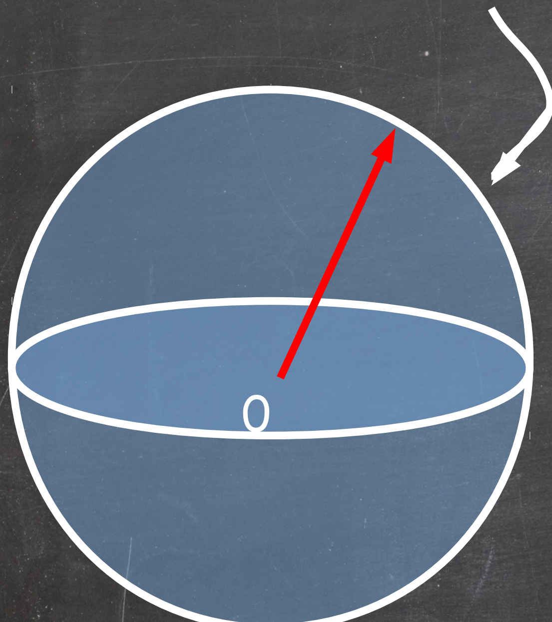

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V_2 = { pairs of perpendicular unit vectors in $\mathbb{R}^3$ }

$= \{ (u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |u|=1, |v|=1, u.v = 0 \}$

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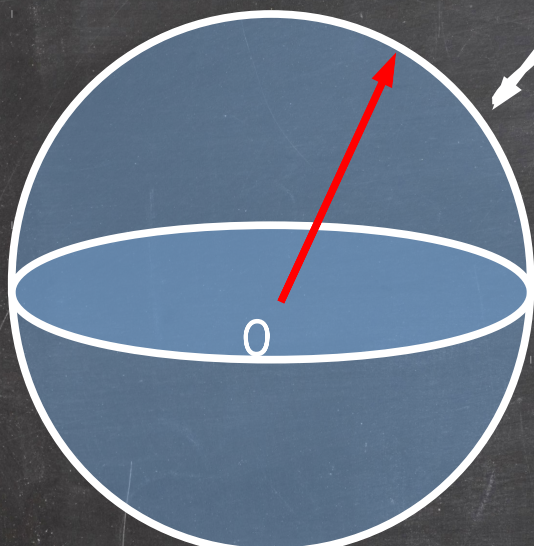

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For each oriented three frame, form the matrix with the corresponding vectors as columns. This is an orthogonal matrix of determinant 1. The set of these matrices is called SO(3).

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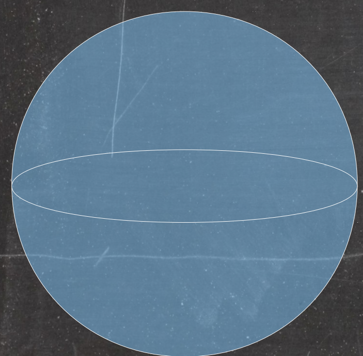

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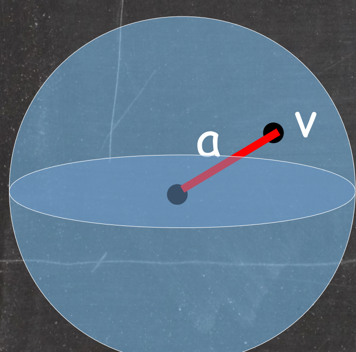

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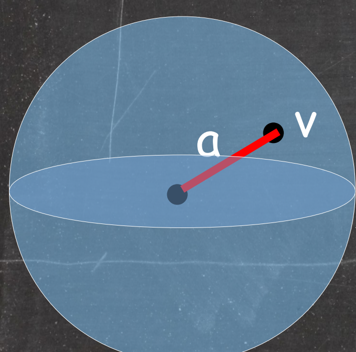

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Therefore, $SO(3) \approx B_\pi / (v = -v) \approx RP^3$

(3 – dim real projective space)

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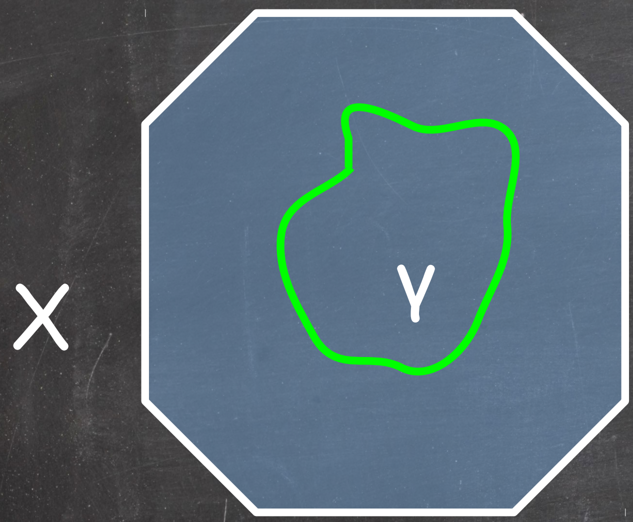

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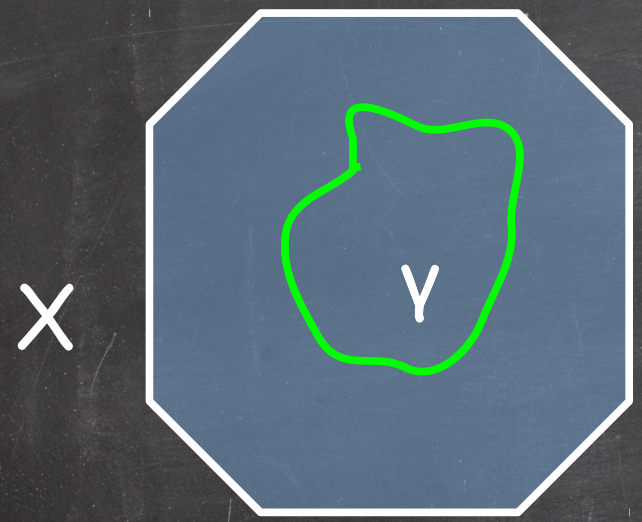

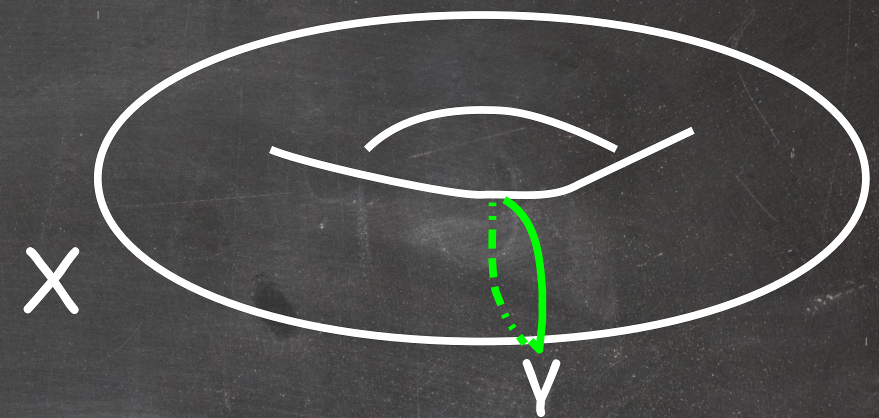

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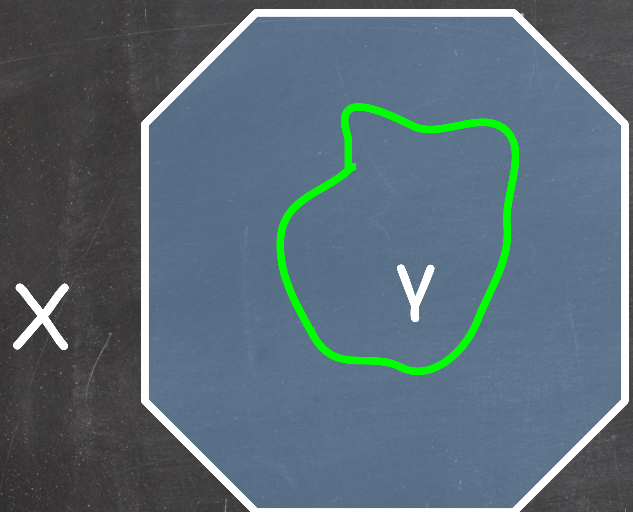

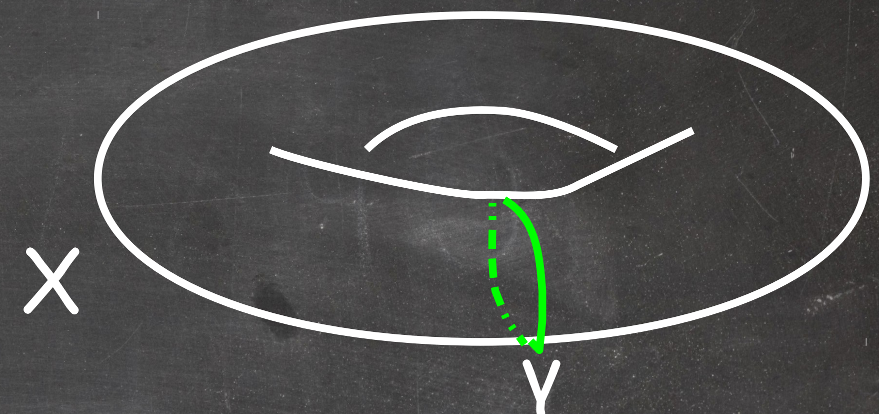

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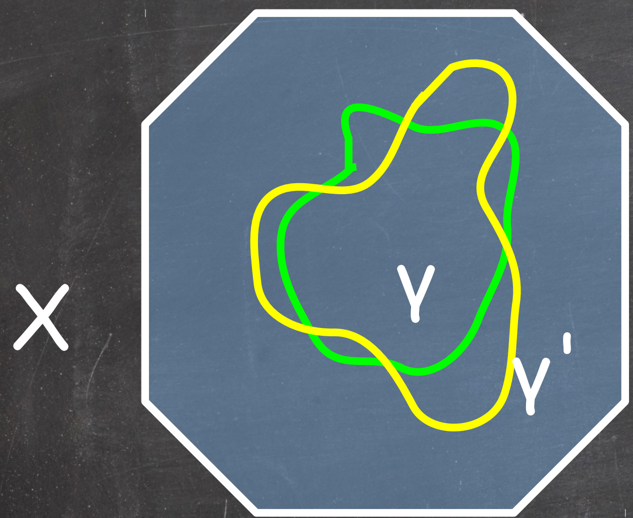

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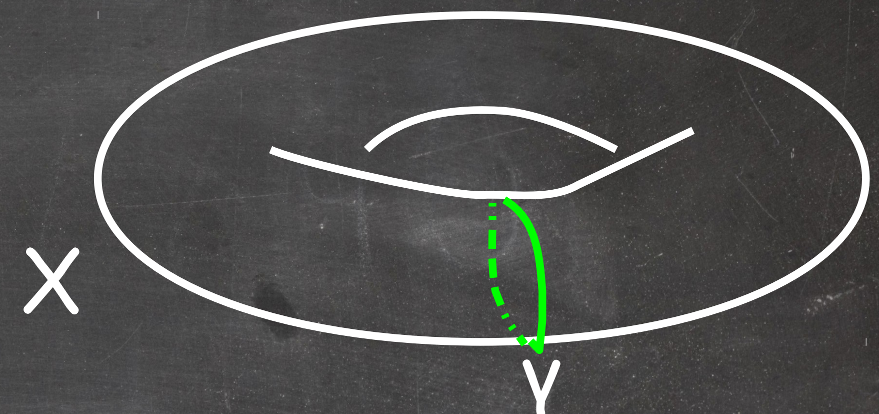

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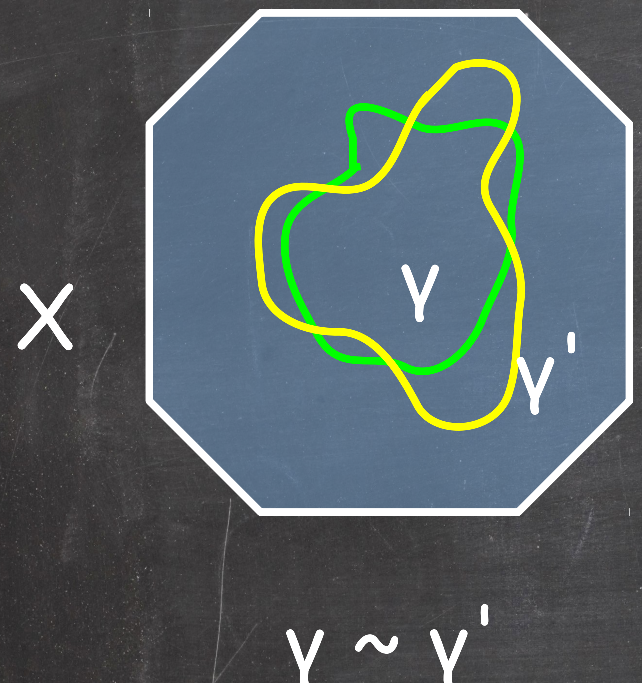

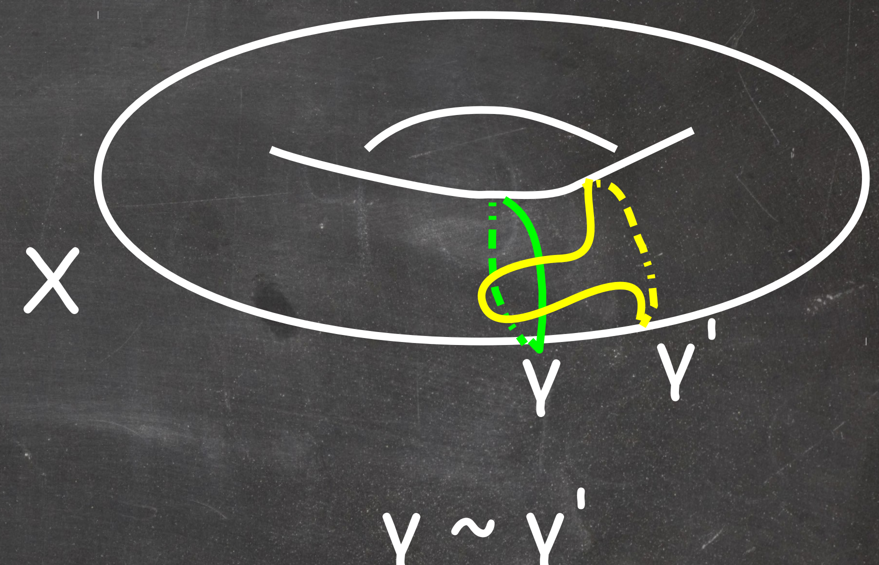

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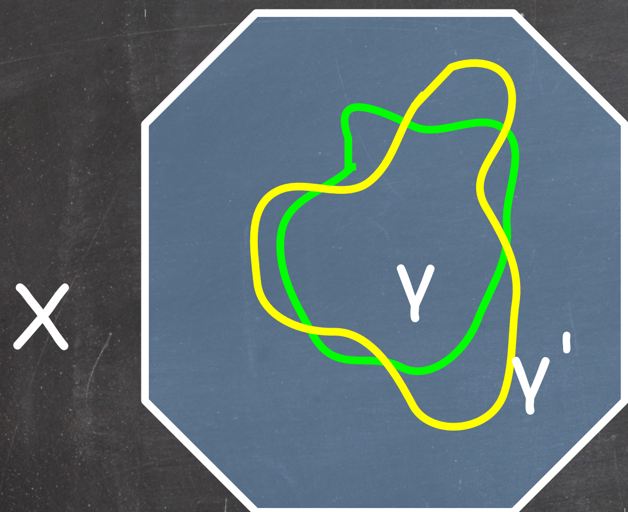

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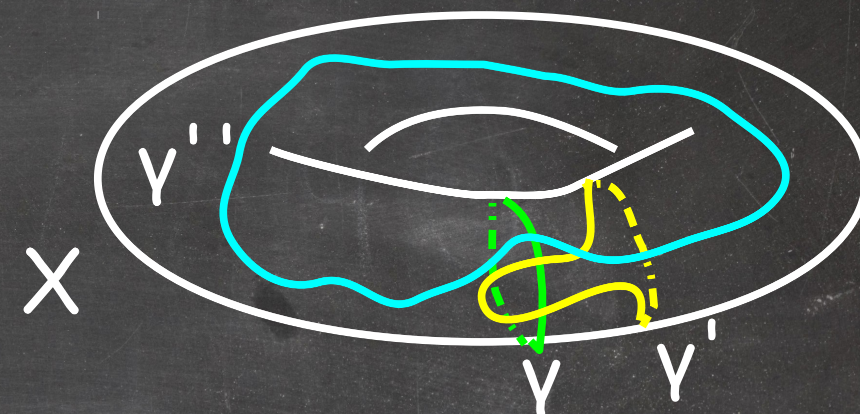

$\gamma \sim \gamma'$ but $\gamma \not\sim \gamma''$

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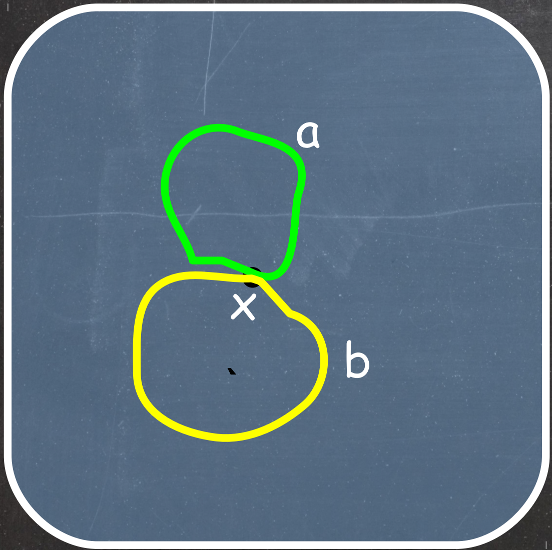

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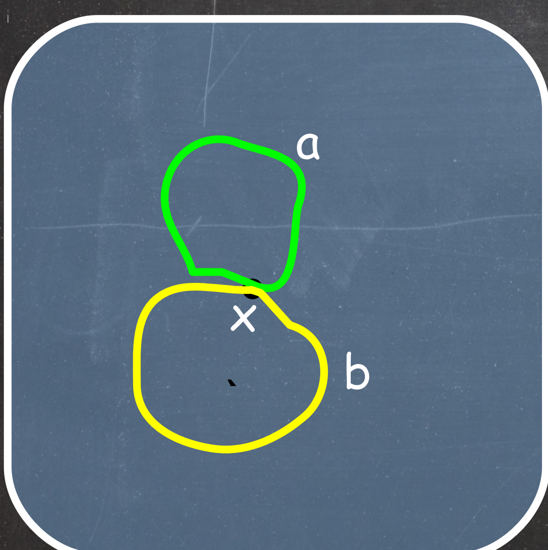

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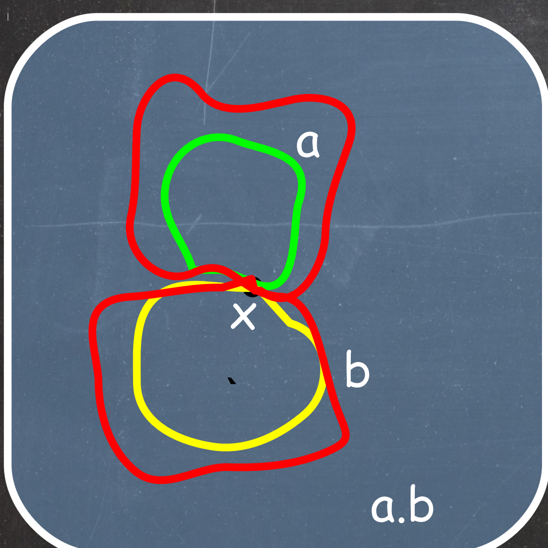

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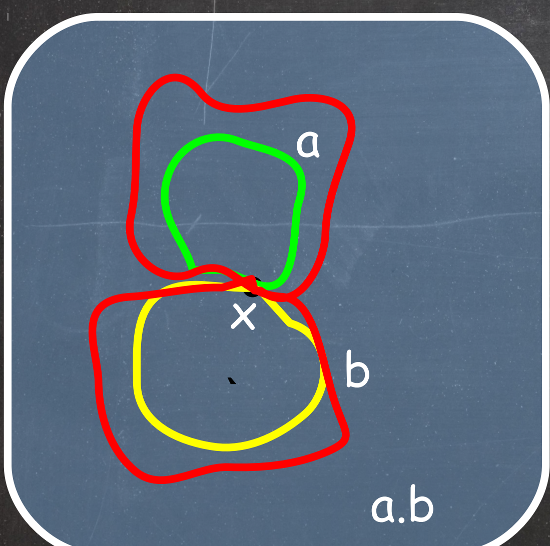

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With this multiplication $\pi_1(X,x)$ becomes a group.

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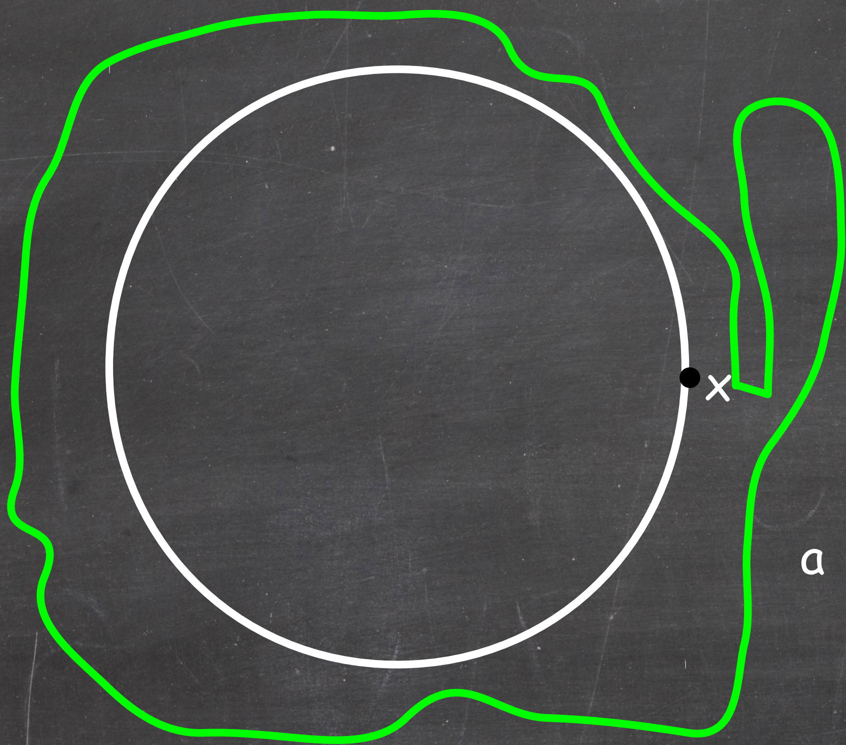

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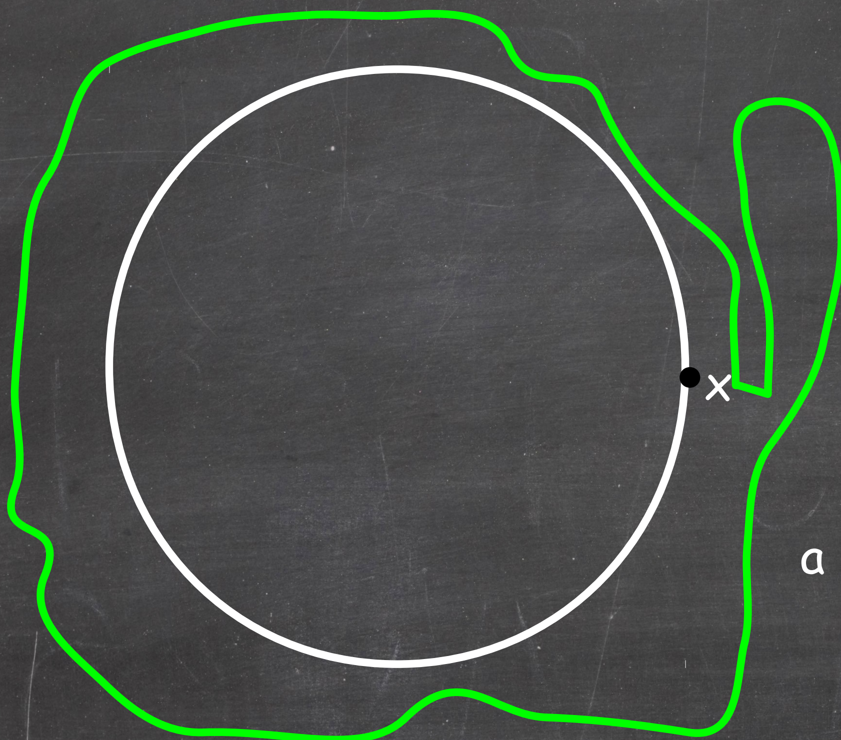

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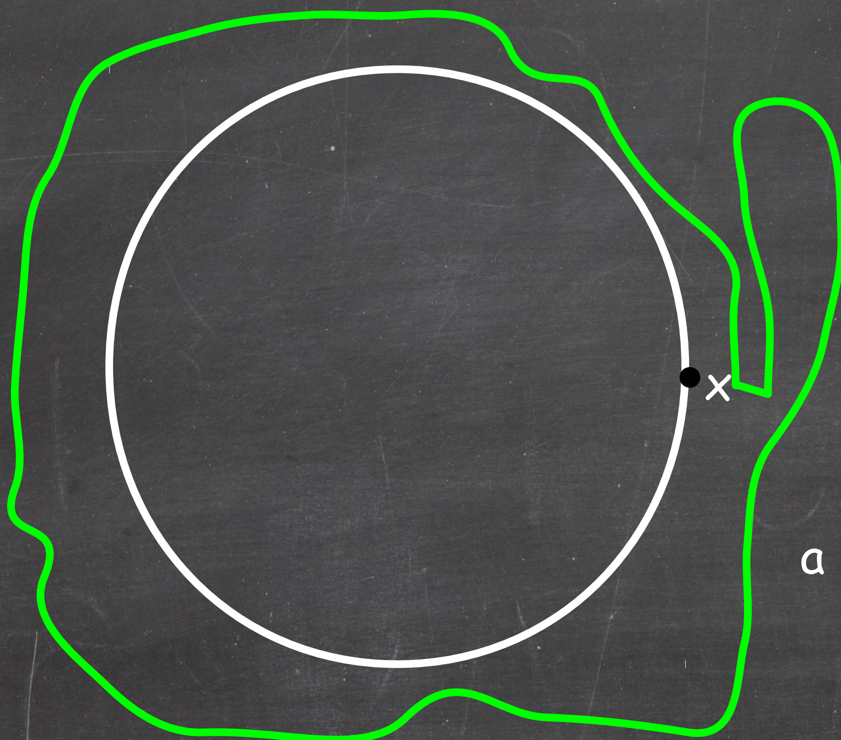

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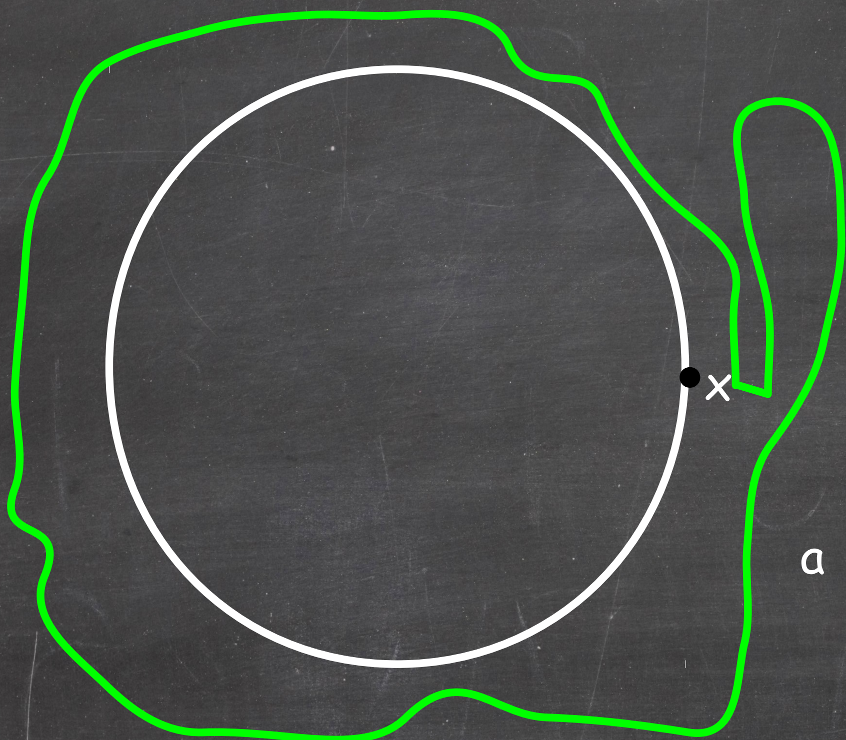

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In fact, two loops are homotopic if and only if they wind around the circle the same number of times in the same direction. This demonstrates :

$$\pi_1(S^1,x) = \mathbb{Z} \text{ (the group of integers)}$$

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A deformation of belt configurations leads to a homotopy of the corresponding loops. Therefore we get an invariant :

$$\{ \text{ Belt configurations } \} \longrightarrow \pi_1(RP^3,x)$$

(x is some point of RP^3)

Recall $RP^3 \approx B_r / (v = -v)$ for some radius r.

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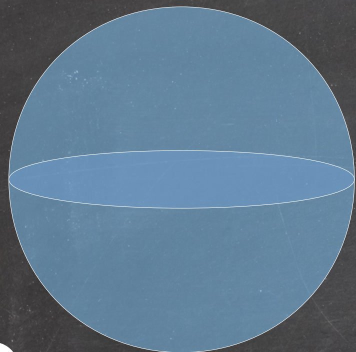

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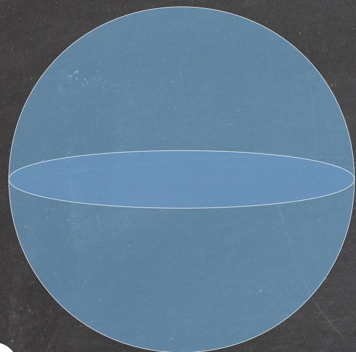

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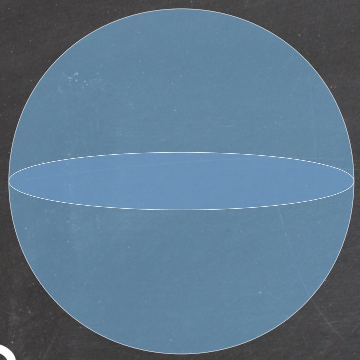

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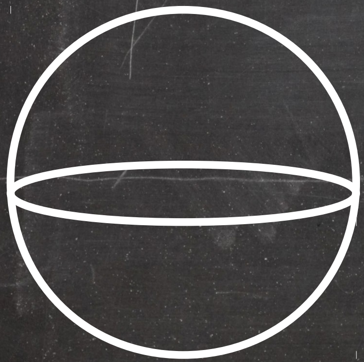
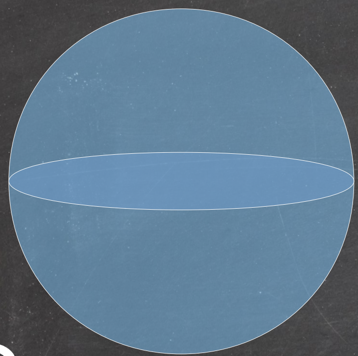


S^2

Recall $RP^3 \approx B_r / (v = -v)$ for some radius r.



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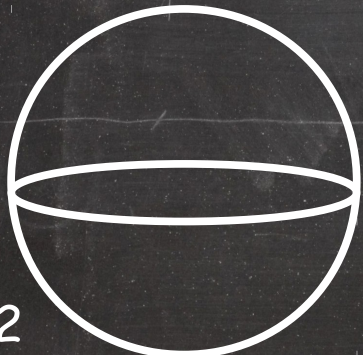

S^2

Similarly any loop can be made to miss a point and such a loop can be homotoped to the equator.

The image of the equator in RP^3 is of the form $S^1/(x=-x)$. The corresponding loop a in RP^3 is the boundary of a disc (image of the sphere). This shows that a is homotopic to constant.

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The image of the half circle in RP^3 is a loop β as the end points map to the same point. This cannot be homotoped to a constant. Also note α = β.β .

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Therefore, $\pi_1(RP^3,x) \approx Z_2$

(the group of integers modulo 2)

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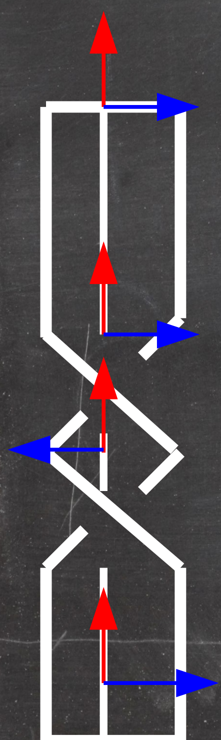
We can compute this to show



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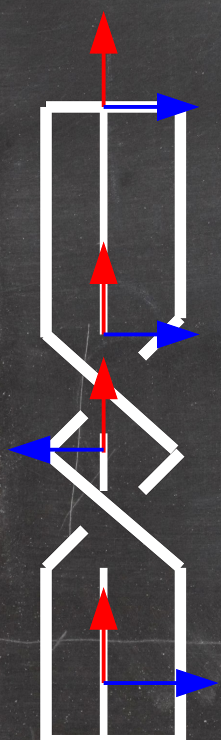

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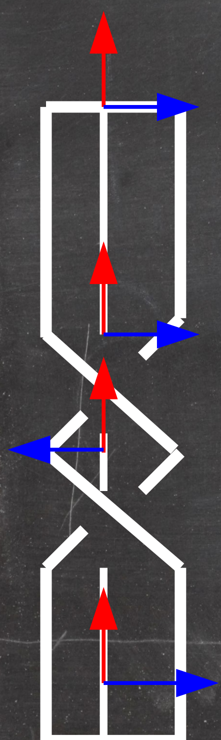
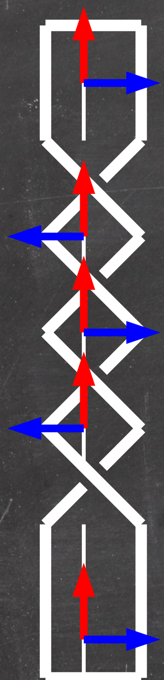
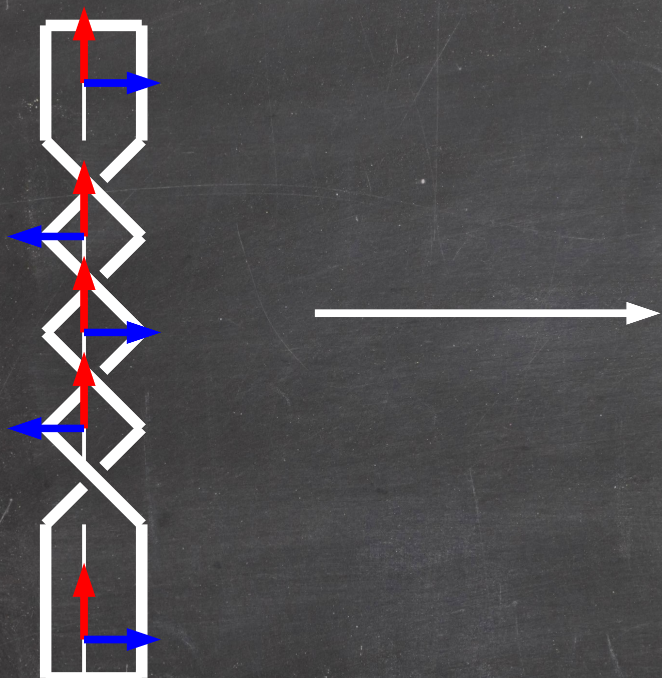


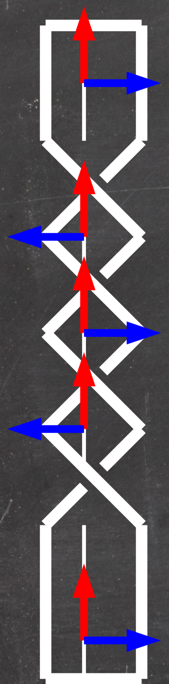
$$\longrightarrow \beta \in \pi_1(RP^3, x)$$

This is a non-trivial element.





$\beta.\beta = \alpha \in \pi_1(RP^3, x)$



$\beta.\beta = \alpha \in \pi_1(RP^3, x)$

This is homotopically trivial.