1. Prerequisites

1.1. **Nets.** A directed set is a preordered (reflexive: $a \preceq a$, transitive: $a \preceq b \preceq c \implies a \preceq c$ set $\mathcal{P}$ such that for any two elements $a, b \in \mathcal{P}, \exists c \in \mathcal{P}$ such that $a \preceq c, b \preceq c$. A net in a topological space X is a map $x : \mathcal{P} \to X$ where $\mathcal{P}$ is a directed set. Nets are often denoted as $\{x_\alpha\}_{\alpha \in \mathcal{P}}$. A net $\{x_\alpha\}$ converges to $x \in X$ if given any open neighborhood U of x there exists α_0 such that for all $\alpha_0 \preceq \alpha, x_\alpha \in U$. A net $\{x_\alpha\}$ in a metric space (X, d) is called Cauchy if $\forall \epsilon > 0, \exists \alpha_0$ such that $d(x_\alpha, x_\beta) < \epsilon$ whenever $\alpha, \beta \succeq \alpha_0$.

Proposition 1.1. *Let X be a topological space and $U \subseteq X$. Then $x \in \overline{U}$ iff there exists a net x_α in U converging to x.*

Proof. Let x_α be a net converging to x. Then given any open neighborhood of x it will contain some x_α and hence an element from U. Thus x is contained in the closure of U. Conversely suppose x is an element from closure of U. Let $\mathcal{P}$ be the directed set consisting of open neighborhoods of x. Define $U_1 \preceq U_2$ if $U_2 \subseteq U_1$. Since x lies in the closure of U every neighborhood V of x will contain an element of U say x_V. This defines a net converging to x. □

Proposition 1.2. *In a complete metric space (X, d) every Cauchy net converges.*

Proof. Exercise. □

1.2. **Sums.** Let X be a set and $a : X \to \mathbb{C}$ be a function. We want to attach a meaning to $\sum_{x \in X} a(x)$. Consider the set $\mathcal{D}$ of finite subsets of X. Given two such finite subsets α, β of X we define $\alpha \preceq \beta$ if $\alpha \subseteq \beta$. With this order $\mathcal{D}$ becomes a directed set. Now consider the net $\{a_\alpha\}_{\alpha \in \mathcal{D}}$, where $a_\alpha = \sum_{x \in \alpha} a(x)$.If this net converges we say that the sum $\sum_{x \in X} a(x)$ is meaningful and $\sum_{x \in X} a(x) = \lim_{\alpha \in \mathcal{D}} a_\alpha$.

Proposition 1.3. *Let $a : X \to \mathbb{C}$ be such that $\sum_{x \in X} a(x)$ makes sense. Then $Supp(a) = \{x \in X : a(x) \neq 0\}$ is countable, and if we fix a one to one and onto map $\phi : \mathbb{N} \to Supp(a)$, then $\sum_{n=1}^{\infty} |a(\phi(n))| < \infty$ and $\sum_{x \in X} a(x) = \sum_{n=1}^{\infty} a(\phi(n))$. Note that this sum does not depend on the map ϕ.*

Proof. Let $\lim_{\alpha \in \mathcal{D}} a_\alpha = A$, that means given $\epsilon > 0$ there exists α_0 such that whenever we have a finite subset α of X such that

$$\alpha \supseteq \alpha_0, |a_\alpha - A| < \epsilon. \tag{1.1}$$

Let $X_n = \{x \in X : \Re(a(x)) > 1/n\}$, then X_n must be finite. Because otherwise, if we take a subset α_k of X_n of size kn, then $\Re(a_{\alpha_k}) > k$. Let $\beta_k = \alpha_k \cup \alpha_0$, then $\Re(a_{\beta_k}) = \Re(a_{\alpha_k}) + \Re(a_{\alpha_0}) > k + \Re(A) - \epsilon$. On the other hand by 1.1, $\Re(a_{\beta_k}) < \Re(A) + \epsilon$, a contradiction. Similarly one shows that $\{x \in X : \Re(a(x)) < -1/n\}$ is finite. Thus we get $\{x \in X : |\Re(a(x)| > 1/n\}$ is finite. Exactly along the same lines one shows that $\{x \in X : |\Im(a(x))| > 1/n\}$ is finite. □

Remark 1.4. *Note that there is nothing special about the Banach space* $\mathbb{C}$*, if* E *is a Banach space and* $a : X \to E$ *is a function we can similarly define* $\sum_{x \in X} a(x)$