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1. Hilbert Spaces

Notation: $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. *Let $\mathcal{H}$ be a vector space. A pre-inner product on $\mathcal{H}$ is a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ such that*

(1) $\langle u, v \rangle = \overline{\langle v, u \rangle}, \forall u, v \in \mathcal{H}$.
(2) $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle, \forall \alpha, \beta \in \mathbb{K}, \forall u, v \in \mathcal{H}$.
(3) $\langle u, u \rangle \geq 0 \ \forall u \in \mathcal{H}$.

Definition 1.2. *A* **Pre-Hilbert Space** *or a pre-inner product space is a pair consisting of vector space along with a pre-inner product.*

Proposition 1.3 (Cauchy-Schwarz Inequality)**.** *Let $\mathcal{H}$ be a vector space equipped with a pre-inner product, then*

$$| < u, v > | \leq \sqrt{\langle u, u \rangle} \sqrt{\langle v, v \rangle}, \forall u, v \in \mathcal{H}.$$

Proof. Let $\langle u, v \rangle = re^{i\theta}, r \geq 0$. Note that if the scalar field is $\mathbb{R}$ then $\theta \in \{\pi, 0\}$. We will divide the proof in cases. The first one is$\langle u, u \rangle = \langle v, v \rangle = 0$.

$$\begin{aligned} 0 &\leq \langle u - e^{-i\theta} v, u - e^{-i\theta} v \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle - e^{-i\theta} \langle u, v \rangle - e^{i\theta} \langle v, u \rangle \\ &= -2r \leq 0. \end{aligned}$$

Thus we get $r = 0$ proving the inequality in this case. Next case is both $\langle u, u \rangle$ and $\langle v, v \rangle$ are not simultaneously zero. Without loss of generality we can assume that $\langle v, v \rangle \neq 0$. Let $t = -\frac{\overline{\langle u, v \rangle}}{\sqrt{\langle v, v \rangle}}$, then,

$$\begin{aligned} 0 &\leq \langle u + tv, u + tv \rangle \\ &= \langle u, u \rangle + |t|^2 \langle v, v \rangle - \frac{2|\langle u, v \rangle|^2}{\langle v, v \rangle} \\ &= \langle u, u \rangle + \frac{|\langle u, v \rangle|^2}{\langle v, v \rangle} - \frac{2|\langle u, v \rangle|^2}{\langle v, v \rangle} \\ &= \langle u, u \rangle - \frac{|\langle u, v \rangle|^2}{\langle v, v \rangle}. \end{aligned}$$

Now transferring $\frac{|\langle u,v\rangle|^2}{\langle v,v\rangle}$ to the other side and multiplying both sides by $\langle v, v\rangle$ we get the result. □

Corollary 1.4. *We have* $\langle u, v\rangle = 0$ *whenever* $\langle v, v\rangle = 0$.

Corollary 1.5. $N = \{v \in \mathcal{H} : \langle v, v\rangle = 0\}$ *is a subspace.*

Proof. Clearly N is closed under scalar multiplication. Only thing we need to show that it is closed under addition. Let $u, v \in N$. Then by the C-S inequality we get $\langle u, v\rangle = 0$. Thus $\langle u + v, u + v\rangle = 0$. □

Corollary 1.6. $\sqrt{\langle u, u\rangle} = \sup_{v:\langle v,v\rangle=1} |\langle u, v\rangle|$

Proof. If $\langle u, u\rangle = 0$ then both sides are zero. Otherwise by the C-S inequality left hand side is less than or equal to right hand side and taking $v = u/\sqrt{\langle u, u\rangle}$ we get the other inequality. □

Definition 1.7. *Let* $\mathcal{H}$ *be a vector space. An inner product on* $\mathcal{H}$ *is a sesquilinear map* $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ *such that*

(1) $\langle\cdot,\cdot\rangle$ *is a pre-inner product.*
(2) *Positive definiteness:* $\langle u, u\rangle = 0 \Longrightarrow u = 0$.

An inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *is a pair consisting of a vector space* $\mathcal{H}$ *along with an inner product on* $\mathcal{H}$

Definition/Proposition 1.8. *Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *be an inner product space, then the map* $\|\cdot\| : \mathcal{H} \to \mathbb{R}_+$ *given by*

$$\|v\| = \begin{cases} \sqrt{\langle v, v\rangle}, v \neq 0 \\ 0, \text{ for } v = 0. \end{cases}$$

is a norm on $\mathcal{H}$. *This norm is referred as the norm associated with the inner product* $\langle\cdot,\cdot\rangle$.

Proof. Let $u, v \in \mathcal{H}$. Only thing we need to verify is $\|u + v\| \leq \|u\| + \|v\|$. That follows from,

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle = \|u\|^2 + \|v\|^2 + 2\Re(\langle u, v\rangle) \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2 \end{aligned}$$

□

Definition 1.9. *An inner product space* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *is called a* **Hilbert space** *if* $\mathcal{H}$ *is complete with respect to the norm associated with the inner product.*

Definition 1.10. *Let* $\mathcal{H}_1, \mathcal{H}_2$ *be Hilbert spaces. A linear map* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *is called* **unitary** *if it is one-to-one, onto and preserves inner products that is,* $\langle Ux, Uy\rangle = \langle x, y\rangle$, *for all* $x, y \in \mathcal{H}_1$. *The Hilbert spaces* $\mathcal{H}_1, \mathcal{H}_2$ *are called unitarily equivalent if there is a unitary* U *from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$.

Proposition 1.11. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces with dense subspaces S_1, S_2 respectively. Let $U : S_1 \to S_2$ be a bijection such that $\langle Ux, Uy \rangle = \langle x, y \rangle$, for all $x, y \in S_1$, then U extends to a unitary map denoted by the same symbol U from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

Proof. Observe that $\|U(x)\| = \|x\|$, for all $x \in \mathcal{S}_1$. Therefore U converts Cauchy sequences to Cauchy sequences. If x is an element in $\mathcal{H}_1$ there is a sequence $\{x_n\}$ of elements of S_1 converging to x. Now $\{U(x_n)\}$ is also Cauchy and therefore converges to some limit. Define Ux as this limit. Clearly this is well defined. By playing the same game with U^{-1} we conclude that the extended map is bijective as well. Continuity of the innerproduct combined with the density of S_i's give $\langle Ux, Uy \rangle = \langle x, y \rangle$, for all $x, y \in \mathcal{H}_1$. □

Definition/Proposition 1.12. *Let $(\mathcal{H}_{pre}, (\cdot,\cdot))$ be a pre-Hilbert space. Let $N = \{v \in \mathcal{H}_{pre} : (v, v) = 0\}$. Then $\langle u + N, v + N \rangle = (u, v)$ defines an inner product on $\mathcal{H}_{pre}/N$. Completion of $\mathcal{H}_{pre}/N$ with respect to the associated norm is called the Hilbert space associated with the pre-Hilbert space $\mathcal{H}_{pre}$.*

Proof. By corollary (1.4) the sesquilinear form $\langle \cdot, \cdot \rangle$ is well defined. Only thing we need to verify is positive definiteness. Let $u \in \mathcal{H}_{pre}$ be such that $\langle u + N, u + N \rangle = (u, u) = 0$. Then $u \in N$ and consequently $u + N = N$. □

Example 1.13. *Let X be a set. Let $\ell_{2,f}(X)$ be the space of $\mathbb{K}$ valued functions on X with finite support. Consider the sesquilinear form on $\ell_{2,f}(X)$ given by, $\langle f, g \rangle = \sum_{x \in X} \overline{f(x)} g(x)$. Note that the sum is finite because both f and g have finite support. The Hilbert space associated with the pre-Hilbert space $(\ell_{2,f}(X), \langle \cdot, \cdot \rangle)$ is denoted by $\ell_2(X)$.*

Proposition 1.14. *Let X be a set. Then $\{f : X \to \mathbb{K} : \sum_{x \in X} |f(x)|^2 < \infty\}$ the space of square summable functions is a Hilbert space with the inner product $\langle f, g \rangle = \sum_{x \in X} \overline{f(x)} g(x)$. This Hilbert space is unitarily equivalent with $\ell_2(X)$.*

Proof. It is immediate that the sesquilinear form $\langle \cdot, \cdot \rangle$ is an inner product. We only need to verify the completeness. Let $\{f_n\}$ be a Cauchy sequence. Then by proposition (4.3) we know that $I = \cup_n Supp(f_n)$ is countable. Now using the completeness of l_2 we conclude that $\{f_n\}$ converges.

Let $S = \{f : X \to \mathbb{K} : Supp(f) \text{ is finite}\}$. It follows from proposition (4.3) that S is dense in the space of square summable functions. The obvious identification between S and $\ell_{2,f}(X)$ preserves inner products and they are dense in their ambient spaces. Therefore we are done by proposition (1.11). □

Theorem 1.15 (Riesz Representation Theorem)**.** *Let $\phi \in \ell_2(X)^*$, then $\exists! \psi \in \ell_2(X)$ with $\|\phi\| = \|\psi\|$ such that $\phi(f) = \langle \psi, f \rangle$. The map $\phi \mapsto \psi$ is a conjugate linear isometry from $\ell_2(X)^*$ to $\ell_2(X)$*

Proof. Without loss of generality assume that $\|\phi\| = 1$. Let $e_x \in \ell_2(X)$ be given by $e_x(y) = \delta_{xy}$, where δ is the Kronecker delta. Then clearly $\|e_x\| = 1$ and $\langle e_x, e_y\rangle = \delta_{xy}$. Now define $\psi : X \to \mathbb{K}$ as $\psi(x) = \overline{\phi(e_x)}$. Given a finite set $F \subseteq X$ define, $\psi_F = \sum_{x\in F} \psi(x)e_x$. Then,

(1) $\|\psi_F\|^2 = \sum_{x\in F} |\psi(x)|^2$.
(2) $\phi(\psi_F) = \sum_{x\in F} |\psi(x)|^2 = \|\psi_F\|^2$.
(3) $\|\psi_F\|^2 = |\phi(\psi_F)| \leq \|\phi\|\|\psi_F\|$, thus $\|\psi_F\| \leq 1$.

As F was arbitrary it follows that $\psi \in \ell_2(X)$. By Cauchy-Schwarz inequality ψ induces a bounded linear functional $\tilde{\psi}$ on $\ell_2(X)$ given by $\tilde{\psi} : f \mapsto \langle \psi, f\rangle$. Then $\tilde{\psi}(e_x) = \overline{\psi(x)} = \phi(e_x)$ and consequently both $\tilde{\psi}, \phi$ agree on the *span*$\{e_x : x \in X\}$. But this is dense in $\ell_2(X)$, hence

$$\phi(f) = \tilde{\psi}(f) = \langle \psi, f\rangle.$$

Uniqueness of ψ follows from the positive definiteness of the inner product. Only thing remaining is, $\|\tilde{\psi}\| = \|\psi\|$. By Cauchy-Schwarz inequality we have $\|\tilde{\psi}\| \leq \|\psi\|$. While the other inequality follows from $|\tilde{\psi}(\psi)| = \|\psi\|^2$. □

Definition 1.16. *Let $\mathcal{H}$ be a Hilbert space. A subset $A \subseteq \mathcal{H}$ is called orthogonal if $x, y \in A, x \neq y$ satisfies $\langle x, y\rangle = 0$. An orthogonal set A is called orthonormal if every $x \in A$ has norm one. An orthonormal basis (o.n.b) is an orthonormal set whose span is dense in $\mathcal{H}$.*

Proposition 1.17. *Let A be an orthonormal set and $f : A \to \mathbb{K}$ be a function such that $\sum_{\alpha\in A} |f(\alpha)|^2 < \infty$, then the net $\{\sum_{\alpha\in F} f(\alpha)\alpha\}_F$ over the directed set of finite subsets of A converges and its limit is denoted by $\sum_{\alpha\in A} f(\alpha)\alpha$. Moreover,*

$$\|\sum_{\alpha\in A} f(\alpha)\alpha\|^2 = \sum_{\alpha\in A} |f(\alpha)|^2. \tag{1.1}$$

Proof. Note that for a finite set $F \subseteq A$,

$$\|\sum_{\alpha\in F} f(\alpha)\alpha\|^2 = \sum_{\alpha\in F} |f(\alpha)|^2. \tag{1.2}$$

By proposition (4.2) it is enough to show that $\{\sum_{\alpha\in F} f(\alpha)\alpha\}_F$ is a Cauchy net. That follows once we show that

$$\forall \epsilon > 0, \exists F_0 \text{ such that } \sum_{\alpha\in F} |f(\alpha)|^2 < \epsilon \text{ whenever } F \cap F_0 = \emptyset.$$

Using the orthonormality of A we get that if $F \subseteq A$ is a finite set with $F \cap F_0 = \emptyset$ then

$$\|\sum_{\alpha\in F} f(\alpha)\alpha\|^2 = \sum_{\alpha\in F} |f(\alpha)|^2 < \epsilon.$$

That is to say that $\{\sum_{\alpha\in F} f(\alpha)\alpha\}_F$ converges. The equality (1.1) follows from the continuity of norm and the equality (1.2). □

Proposition 1.18. *Let A be an orthonormal set in a Hilbert space $\mathcal{H}$ and $v \in \mathcal{H}$. Then the sum $\sum_{\alpha\in A} \langle \alpha, v\rangle\alpha$ converges. If we call the limit w then $\|w\| \leq \|v\|$ and $\langle v - w, \alpha\rangle = 0$ for all $\alpha \in A$.*

Proof. Let $F \subseteq A$ be a finite subset. Then, $\{v - \sum_{\alpha\in F}\langle\alpha, v\rangle\alpha\} \cup F$ is an orthogonal set and $v = (v - \sum_{\alpha\in F}\langle\alpha, v\rangle\alpha) + \sum_{\alpha\in F}\langle\alpha, v\rangle\alpha$. Thus,

$$\|v\|^2 = \sum_{\alpha\in F} |\langle\alpha, v\rangle|^2 + \|v - \sum_{\alpha\in F}\langle\alpha, v\rangle\alpha\|^2. \tag{1.3}$$

Therefore,

$$\sup_F \sum_{\alpha\in F} |\langle\alpha, v\rangle|^2 \leq \|v\|^2. \tag{1.4}$$

Proposition (1.17) allows us to conclude that the sum $\sum_{\alpha\in A} \langle\alpha, v\rangle\alpha$ converges. Let w be the limit. Continuity of norm along with (1.4) gives $\|w\| \leq \|v\|$.

Let $\alpha' \in A$. By CS-inequality we know that inner product is a continuous map. Therefore

$$\langle v - w, \alpha'\rangle = \lim_F \langle v - \sum_{\alpha\in A} \langle\alpha, v\rangle\alpha, \alpha'\rangle = 0,$$

because $\langle v - \sum_{\alpha\in A} \langle\alpha, v\rangle\alpha, \alpha'\rangle = 0$ whenever $\alpha' \in F$. □

Definition 1.19 (Gram-Schmidt Orthogonalization)**.** *Let F be a finite orthonormal set in a Hilbert space $\mathcal{H}$ and $v \in \mathcal{H} \setminus Span\{u : u \in F\}$. Then*

$$GS(v; F) = \{v - \sum_{\alpha\in F}\langle\alpha, v\rangle\alpha\} \cup F$$

is an orthogonal set called the result of applying Gram-Schmidt orthogonalization to the pair$(v; F)$.

Definition 1.20. *Let μ be a finite measure on $\mathbb{R}$ such that $\int |x|^n d\mu(x) < \infty, \forall n \geq 0$. Then consider the Hilbert space $L_2(\mathbb{R}, \mu)$ and successively define polynomials P_n*

as follows,

$$\begin{aligned}
p_0(x) &= 1 \\
P_0 &= \frac{p_0}{\|p_0\|_2} \\
p_1(x) &= x - (\int_{\mathbb{R}} yP_0(y)d\mu(y))P_0(x) \\
P_1 &= \frac{p_1}{\|p_1\|_2} \\
. \quad . &\quad \cdots \\
p_n(x) &= x^n - \sum_{j=0}^{n-1}(\int_{\mathbb{R}} y^n P_j(y)d\mu(y))P_j(x) \\
P_n &= \frac{p_n}{\|p_n\|_2} \\
. \quad . &\quad \cdots
\end{aligned}$$

These polynomials are called orthogonal polynomials with respect to the measure μ.

Corollary 1.21. *In the notation of proposition (1.18),*

$$\|v\|^2 = \|w\|^2 + \|v - w\|^2.$$

Proof. Note that w belongs to the closed span of A. The continuity of the inner product gives $\langle v - w, w\rangle = 0$ which in turn gives the result. □

Corollary 1.22 (Abstract Fourier Expansion/ Parseval relations)**.** *Let A be an o.n.b, then every $v, v' \in \mathcal{H}$ satisfies,*

$$v = \sum_{\alpha \in A} \langle \alpha, v\rangle \alpha. \tag{1.5}$$

$$\langle v', v\rangle = \sum_{\alpha \in A} \langle v', \alpha\rangle \cdot \langle \alpha, v\rangle \tag{1.6}$$

The expansion (1.5) is called the abstract Fourier expansion with respect to the basis A.

Proof. Let $w = \sum_{\alpha \in A} \langle \alpha, v\rangle \alpha$, then by proposition (1.18) $\langle v - w, \alpha\rangle = 0$ for every α from the span of A. But span of A is dense in $\mathcal{H}$, thus $\langle v - w, v'\rangle = 0$ for all $v' \in \mathcal{H}$. Taking $v' = v - w$ we get $v = w$.

To see (1.6) observe that by the continuity of the inner product we have

$$\begin{aligned}
\langle v', v\rangle &= \lim_F \langle \sum_{\alpha \in F} \langle \alpha, v'\rangle \alpha, \sum_{\alpha \in F} \langle \alpha, v\rangle \alpha\rangle \\
&= \lim_F \sum_{\alpha \in F} \langle v', \alpha\rangle \cdot \langle \alpha, v\rangle \\
&= \sum_{\alpha \in A} \langle v', \alpha\rangle \cdot \langle \alpha, v\rangle.
\end{aligned}$$

□

Proposition 1.23. *Every orthonormal set in a Hilbert space can be extended to an orthonormal basis.*

Proof. Let B be an orthonormal set. Consider the partially ordered set

$$\mathcal{P} = \{A \subseteq \mathcal{H} : A \text{ is an orthonormal set that contains} B\}$$

ordered by inclusion. By Zorn's lemma obtain a maximal element, say A. We wish to show that A is an o.n.b. If it is not, then there exists a $v \in \mathcal{H} \setminus \overline{SpanA}$. Let $w = \sum_{\alpha \in A} \langle \alpha, v\rangle\alpha$, then $\|v - w\| \neq 0$ because otherwise $v(= w)$ becomes an element of the closed span of A. By proposition (1.18) $\{\frac{(v-w)}{\|v-w\|}\} \cup A$ becomes an orthonormal set containing A. This contradicts the maximality of A. □

Corollary 1.24 (Projection Theorem)**.** *Let $\mathcal{H}' \subseteq \mathcal{H}$ be a closed subspace. Let $v \in \mathcal{H}$, then there exists unique $w \in \mathcal{H}'$ such that*

$$\|v - w\| = \min\{\|v - x\| : x \in \mathcal{H}'\}.$$

The association $v \mapsto w$ defines a linear map $P : \mathcal{H} \to \mathcal{H}$ such that $P^2 = P$, range of P is $\mathcal{H}'$ and $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in \mathcal{H}$. The map P is called the orthogonal projection onto $\mathcal{H}'$.

Proof. Let A' be an o.n.b of $\mathcal{H}'$. Extend it to A, an o.n.b for $\mathcal{H}$. Let $v \in \mathcal{H}$. Let $v \in \mathcal{H}$. Then by corollary (1.22), $v = \sum_{\alpha\in A}\langle \alpha, v\rangle\alpha$. Let $x \in \mathcal{H}'$, then again by corollary (1.22) we get $x = \sum_{\alpha\in A'}\langle \alpha, v\rangle\alpha$. Proposition (1.17) gives,

$$\begin{aligned}
\|v - x\|^2 &= \Big\|\sum_{\alpha\in A}\langle \alpha, v\rangle\alpha - \sum_{\alpha\in A'}\langle \alpha, x\rangle\alpha\Big\|^2 \\
&= \sum_{\alpha\in A\setminus A'} |\langle \alpha, v\rangle|^2 + \sum_{\alpha\in A'} |\langle \alpha, v\rangle - \langle \alpha, x\rangle|^2 \\
&\geq \sum_{\alpha\in A\setminus A'} |\langle \alpha, v\rangle|^2.
\end{aligned}$$

The last inequality becomes a equality iff $\langle \alpha, v\rangle = \langle x, \alpha\rangle, \forall \alpha \in A'$. That is to say that the minimization problem is solved by $w = \sum_{\alpha\in A'}\langle \alpha, v\rangle\alpha$. This expression also shows that P is a linear map and $P^2 = P$. Let $x, y \in \mathcal{H}$, then

$$\begin{aligned}
\langle Px, y\rangle &= \langle \sum_{\alpha\in A'} \langle \alpha, x\rangle\alpha, y\rangle \\
&= \sum_{\alpha\in A'} \langle x, \alpha\rangle\langle \alpha, y\rangle \\
&= \langle x \sum_{\alpha\in A'} \langle \alpha, y\rangle\alpha\rangle \\
&= \langle x, Py\rangle.
\end{aligned}$$

□

Corollary 1.25 (Corollary to Projection Theorem)**.** *Let $\mathcal{H}_1 \subseteq \mathcal{H}$ be a closed subspace. Then*
$$\mathcal{H}_1^\perp := \{u \in \mathcal{H} : \langle u, v\rangle = 0, \forall v \in \mathcal{H}_1\}$$
is a closed subspace called the orthocomplement of $\mathcal{H}_1$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$.

Proof. The orthocomplement is closed because if $u_n \in \mathcal{H}_1^\perp$ and $\|u_n - u\| \to 0$, then given any $v \in \mathcal{H}_1$ using the continuity of the innerproduct (a consequence of the Cauchy-Schwarz inequality) we get
$$\langle u, v\rangle = \lim_{n\to\infty} \langle u_n, v\rangle = 0.$$
We only need to show that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. Let P be the orthogonal projection onto $\mathcal{H}_1$. Then $\mathcal{H}_1 = \{v \in \mathcal{H} : Pv = v\}$. Let $u \in \mathcal{H}, v = P(u) \in \mathcal{H}_1$. Then given any $w = P(w) \in \mathcal{H}_1$,
$$\begin{aligned}\langle u - P(u), w\rangle &= \langle (I-P)(u), P(w)\rangle, \text{ where } I : u \mapsto u,\\ &= \langle P(I-P)(u), w\rangle\\ &= \langle (P-P)(u), P(w)\rangle, [\text{ since } P = P^2]\\ &= 0.\end{aligned}$$

Therefore $u - P(u) \in \mathcal{H}_1^\perp$. Thus $u = P(u) + (u - P(u)) \in \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. □

Corollary 1.26. *Let A be an o.n.b in a Hilbert space $\mathcal{H}$. There is a unitary $U : \mathcal{H} \to \ell_2(A)$.*

Proof. Recall that by proposition (1.14),
$$\ell_2(A) \cong \{f | f : A \to \mathbb{K}, \sum_{\alpha\in A} |f(\alpha)|^2 < \infty\}.$$
Define $U(v) : A \to \mathbb{K}$ as $U(v)(\alpha) = \langle \alpha, v\rangle$, then corollary (1.22) shows that U is an inner product preserving map. Proposition (1.17) shows that U is onto, hence a unitary. □

Corollary 1.27 (Riesz Representation Theorem)**.** *Let $\phi \in \mathcal{H}^*$, then $\exists!\psi \in \mathcal{H}$ with $\|\phi\| = \|\psi\|$ such that $\phi(f) = \langle \psi, f\rangle$. The map $\phi \mapsto \psi$ is a conjugate linear isometry from $\mathcal{H}^*$ to $\mathcal{H}$*

Proof. Let A be an o.n.b and $U : \ell_2(A) \to \mathcal{H}$ be the unitary defined in the previous corollary. Let $\phi' \in \ell_2(A)^*$ be the functional $v \mapsto \phi(U(v))$. Then by theorem (1.15) we can obtain $\psi' \in \ell_2(A)$ such that $\phi(U(v)) = \phi'(v) = \langle \psi', v\rangle$. Let $\psi = U\psi'$. Then,
$$\phi(v) = \phi(U(U^{-1}v) = \langle \psi', U^{-1}(v)\rangle = \langle \psi, v\rangle.$$
Also $\|\psi\| = \|\psi'\| = \|\phi'\| = \|\phi\|$. □

Exercise 1.28. *A bounded linear map U on a Hilbert space is a unitary iff $U^*U = UU^* = I$, where I stands for the identity operator.*

Proposition 1.29. *Any two o.n.b have same cardinality.*

Proof. Let $\mathcal{H}$ be a Hilbert space with two orthonormal basis A, B. We will prove the proposition in the infinite dimensional case only. Fix a countable dense subset $\mathbb{K}'$ of $\mathbb{K}$. Let,

$$\mathcal{H}_A = \{v \in \mathcal{H} | \{a \in A : \langle v, a\rangle \neq 0\} \text{ is finite and } \langle v, a\rangle \in \mathbb{K}', \forall a \in A\}$$

Then $\mathcal{H}_A$ is dense in $\mathcal{H}$ and is in bijection with $\cup_{n=1}^{\infty} A^n \times \mathbb{K}'^n$ which is in bijection with A. Define $f : B \to \mathcal{H}_A$, such that $\|b - f(b)\| < 1/8$, for all $b \in B$. Orthonormality of B implies $\|b - b'\| > 1$ whenever we have two distinct elements of B. Thus given any two distinct elements $b, b' \in B$ we have $\|f(b) - f(b')\| > 1/2$. That is to say that f is one to one. This shows that the cardinality of A is greater than or equal to that of B. By symmetry we get the other inequality and conclude both A and B have the same cardinality. □

Proposition 1.30. *Let $\mathcal{H}$ be a separable Hilbert space. Then any o.n.b is countable.*

Proof. Fix a countable dense set S. Let A be an o.n.b. Define a function $f : A \to S$ such that $\|f(\alpha) - \alpha\| < 1/2$. Then f is one to one because, given any two distinct α, α' of A, we have

$$\|f(\alpha) - f(\alpha')\| \geq \|\alpha - \alpha'\| - \|f(\alpha) - \alpha\| - \|f(\alpha') - \alpha'\| > 1 - 1/2 - 1/2 = 0.$$

This shows that A is countable. □

2. A Typical Business in Hilbert Space Theory

In this section and the rest of this chapter we will assume our ground field is $\mathbb{C}$. We have seen that Hilbert spaces are classified by a cardinal namely that of an onb. This may give the impression that this ends the theory of Hilbert spaces. But that is not the case because Hilbert spaces often come equipped with extra structures like representations of other algebraic structures and one often looks for unitary equivalences that respect these structures. Now we will illustrate this with the first and simplest such result. None the less we will see that this result has significant applications. Let us start with the simplest locally compact abelian group, namely $\mathbb{Z}$. Every locally compact abelian group G determines another such group denoted by $\widehat{G}$ as follows.

$$\widehat{G} = \{\xi | \xi : G \to \mathbb{T} \text{ is a group homomrphism }\},$$

where $\mathbb{T}$ is the unit circle. A homomorphism from G to $\mathbb{T}$ is called a **character** on G. Thus $\widehat{G}$ is the space of characters. The group operations of $\widehat{G}$ are defined as $\xi^{-1}(g) = \xi(g)^{-1}$ and $(\xi_1 \cdot \xi_2)(g) = \xi_1(g)\xi_2(g)$. The group $\widehat{G}$ is called the Pontryagin dual of G. Give $\widehat{G}$ the topology of uniform convergence on compact sets. We will prove later that with this topology $\widehat{G}$ becomes a locally compact topological group. Since a group homomorphism from $\mathbb{Z}$ is specified by its action on the generator we can identify $\widehat{\mathbb{Z}}$ as a set with $\mathbb{T}$ itself. It is also easy to see that the way we defined the topology on $\widehat{G}$, the resulting topology on $\mathbb{T}$ is the usual topology. We will identify it with the quotient space $\mathbb{R}/\mathbb{Z}$. Thus we will parametrize points on the circle by $\{e^{2\pi i\theta} : \theta \in [0,1]\}$. By its translation invariance Lebesgue measure induces a measure on $\mathbb{T}$. This measure will be denoted by $d\theta$. The Borel sigma algebra on $\mathbb{T}$ is countably generated hence $L_2(\mathbb{T})$ is separable. We know that any two separable Hilbert spaces have countable o.n.b, hence they are unitarily equivalent. Thus $L_2(\mathbb{T})$ is equivalent with $\ell_2(\mathbb{Z})$. But the question is, are their "better unitaries". So, what do we mean by a "better" unitary. We need to work a little to make sense of this. Observe that given a Hilbert space $\mathcal{H}$, unitary operators on $\mathcal{H}$, denoted by $U(\mathcal{H})$ is a group. We have a group homomorphism $U : \mathbb{Z} \to U(\ell_2(\mathbb{Z}))$ given by $U(n)\xi : m \mapsto \xi(m-n)$. Using this map we can produce a map from $\pi_{\mathbb{Z}} : \ell_1(\mathbb{Z}) \to B(\ell_2(\mathbb{Z}))$ as follows,

$$\pi_{\mathbb{Z}}(f) = \sum f(n)U(n), \text{ where } f \in \ell_1(\mathbb{Z}).$$

Note that

$$\|\pi_{\mathbb{Z}}(f)\| \leq \sum |f(n)| \|U(n)\| = \|f\|_1. \tag{2.7}$$

We know that $\ell_1(\mathbb{Z})$ is a Banach Space, has it got some more algebraic structure? Yes, it is an algebra (The words algebra and ring are synonymous, even though the former is preferred by analysts.) provided we define the product of $f, g \in \ell_1(\mathbb{Z})$ as

$$(f \star g)(k) = \sum_n f(n)g(k-n).$$

Of course we need to verify that the sequence $\{(f \star g)(k)\} \in \ell_1(\mathbb{Z})$ and that follows from,

$$\begin{aligned}
\sum_k |(f \star g)(k)| &= \sum_k |\sum_n f(n)g(k-n)| \\
&\leq \sum_k \sum_n |f(n)||g(k-n)| \\
&= \sum_n \sum_k |f(n)||g(k-n)|, \quad \text{(Tonneli in action here)}, \\
&\leq \|f\|_1 \|g\|_1.
\end{aligned}$$

We have not only shown that $(f \star g) \in \ell_1(\mathbb{Z})$, we have also shown that $\|f \star g\|_1 \leq \|f\|_1 \|g\|_1$. In other words we have shown that $\ell_1(\mathbb{Z})$ is a Banach Algebra. Since we have already used the term we may as well define this concept formally.

Definition 2.1. *A Banach algebra $\mathcal{A}$ is a Banach space along with an algebra/ ring structure such that $\|a \cdot b\| \leq \|a\| \cdot \|b\|, \forall a, b \in \mathcal{A}$.*

There is nothing special about $\mathbb{Z}$. The same argument goes verbatim in general.

Definition 2.2. *Let G be a locally compact group. A measure λ on G is called left(right) invariant if $\lambda(g.B) = \lambda(B)(\lambda(B.g) = \lambda(B))$ for each Borel subset B of G. This is equivalent with*

$$\int_G f(g \cdot h) d\lambda(h) = \int_G f(h) d\lambda(h), \forall f \in L_1(G, \lambda). \tag{2.8}$$

Theorem 2.3. *Let G be a locally compact Hausdorff topological group. Then G admits a left (right) invariant measure λ and it is unique upto scaling by a positive real number. Such a measure is called a left (right) Haar measure. If we just say Haar measure we mean left Haar measure.*

We will not prove this in this generality. For a compact second countable group we will establish this later.

Proposition 2.4. Let G be a locally compact group and λ be a Haar measure on G. Then $\mathcal{A} = L_1(G, \lambda)$ is a Banach algebra with multiplication defined by

$$(f_1 \star f_2) = \int f_1(g) f_2(g^{-1}h) d\lambda(g).$$

This multiplication is called **Convolution**.

Proposition 2.5. *Convolution is associative.*

Proof. (1) $f_1 \star f_2 \in L_1$:

$$\begin{aligned} \int |f_1 \star f_2(h) d\lambda(h) &\leq \int\int |f_1(g)||f_2(g^{-1}h| d\lambda(g) d\lambda(h) \\ &= \int |f_1(g)| d\lambda(g) \int |f_2(h)| d\lambda(h) \\ &= \|f_1\|_1 \|f_2\|_1 \end{aligned}$$

Therefore we have proved

$$\begin{aligned} f_1 \star f_2 &\in L_1(G)\text{and} \\ \|f_1 \star f_2\|_1 &\leq \|f_1\|_1 \|f_2\|_1 \end{aligned}$$

(2) $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$:

$$\begin{aligned} (f_1 \star f_2) \star f_3(u) &= \int (f_1 \star f_2)(v) f_3(v^{-1}udv \\ &= \int\int f_1(w) f_2(w^{-1}v) f_3(v^{-1}u) dwdv \\ &= \int\int f_1(w) f_2(v) f_3(v^{-1}w^{-1}u) dwdv \\ &= f_1 \star (f_2 \star f_3)(u) \end{aligned}$$

□

The space $\widehat{G}$ determines another Banach algebra $C_b(\widehat{G})$ provided we define product of two functions as their pointwise product. Of course in the present case $\widehat{G}$ itself is compact, hence $C_b(\widehat{G}) = C(\widehat{G})$. We have an obvious homomorphism $\widetilde{\pi}_{\widehat{G}} : C_b(\widehat{G}) \to U(L_2(\widehat{G}))$ given by $\widetilde{\pi}_{\widehat{G}}(f)\xi = f \cdot \xi$ where $f \cdot \xi$ is the square integrable function on $\widehat{G}$, $x \mapsto f(x)\xi(x)$. The obvious inequality

$$\int_{\widehat{G}} |f \cdot \xi|^2 d(Haar) \leq \sup_{x \in \widehat{G}} |f(x)| \int_{\widehat{G}} |\xi|^2 d(Haar),$$

shows that

$$\|\widetilde{\pi}_{\widehat{G}}(f)\| \leq \|f\|. \tag{2.9}$$

Thus $\widetilde{\pi}_{\widehat{G}}$ is a Banach algebra homomorphism. The final ingredient is a transform, probably the most celebrated with in the whole of Mathematics called Fourier transform. This is a homomorphism $\mathcal{F}_{\mathbb{Z}} : \ell_1(\mathbb{Z}) \to C(\mathbb{T})$, given by $\mathcal{F}(f)(e^{2\pi i\theta}) = \sum_n f(n)e^{-2\pi in\theta}$. Absolute summability of the sequence $\{f(n)\}$ ensures uniform convergence of the right hand side. In the general case of a locally compact abelian group G, the Fourier transform $\mathcal{F}_G : L_1(G) \to C_b(\widehat{G})$ is defined as

$$\mathcal{F}_G(f)(\xi) = \int_G \xi(g^{-1}) f(g) dg. \tag{2.10}$$

$$\|\mathcal{F}_G(f)\| = \sup_{\xi \in \widehat{G}} |\mathcal{F}_G(f)| \leq \int_G |\xi(g^{-1}) f(g)| dg = \|f\|_1 \tag{2.11}$$

Of course we need to establish that the right hand side defines a bounded continuous function. In fact more is true, Fourier transform actually lands in the space of continuous functions vanishing at infinity. This is often referred as the Riemann-Lebesgue lemma and will be taken up during our discussion of Banach algebras. At this moment we will just show that it defines a bounded continuous function. However $\widehat{\mathbb{Z}}$ itself being compact $C(\widehat{\mathbb{Z}})$ coincides with $C_0(\widehat{\mathbb{Z}})$.

Proposition 2.6. *Let $f \in L_1(G)$, then $\mathcal{F}(f) \in C_b(\widehat{G})$.*

Proof. Let C be a Borel subset of G contained in a compact set K. The indicator function of C denoted by 1_C is defined as

$$1_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following

(1) The span of $\{1_C : C \text{ contained in a compact set } \}$ is dense in $L_1(G)$.
(2) For such a C, by an easy application of the dominated convergence theorem $\mathcal{F}_G(1_C)$ is a continuous function.
(3) The estimate $\|\mathcal{F}_G(f)\| \leq \|f\|_1$ shows that for an arbitrary f, $\mathcal{F}_G(f)$ is a uniform limit of a sequence of bounded continuous functions. Hence $\mathcal{F}_G(f)$ is bounded and continuous.

□

Proposition 2.7. *Fourier transform is a homomorphism.*

Proof. Let $f_1, f_2 \in L_1(G)$. Then,

$$\begin{aligned} \mathcal{F}_G(f_1 \star f_2)(\xi) &= \int_G \overline{\xi(g)}(f_1 \star f_2)(g)dg \\ &= \int_G \int_G \overline{\xi(h)\xi(h^{-1}g)} f_1(h) f_2(h^{-1}g)dhdg \\ &= \mathcal{F}_G(f_1)(\xi)\mathcal{F}_G(f_2)(\xi). \end{aligned}$$

Thus

$$\mathcal{F}_G(f_1 \star f_2) = \mathcal{F}_G(f_1) \cdot \mathcal{F}_G(f_2)$$

□

Now we can state what do we mean by a "good" unitary from $\ell_2(\mathbb{Z})$ to $L_2(\mathbb{T})$. A unitary $U : \ell_2(\mathbb{Z}) \to L_2(\mathbb{T})$ is called "good" if given any element f of $\ell_1(\mathbb{Z})$, we have

$$U\pi_{\mathbb{Z}}(f) = \widetilde{\pi}_{\mathbb{T}}(\mathcal{F}_{\mathbb{Z}}(f))U. \tag{2.12}$$

This is also expressed by saying that U **intertwines the representations** $\pi_{\mathbb{Z}}$ **and** $\widetilde{\pi}_{\mathbb{T}} \circ \mathcal{F}_{\mathbb{Z}}$.

Theorem 2.8. *The Fourier transform $\mathcal{F}_{\mathbb{Z}}$ extends to a "good" unitary.*

Proof. Let $\delta_n \in \ell_1(\mathbb{Z})$ be the map

$$\delta_n(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Let c_{00} be the span of δ_n's for $n \in \mathbb{Z}$. Note that these δ_n's can also be considered as elements of $\ell_2(\mathbb{Z})$ and strictly speaking when considered as elements of $\ell_2(\mathbb{Z})$

one should use different symbols for them. But we won't do that and use the same symbol. Also consider the maps $z_n \in C(\mathbb{T})$ given by $e^{2\pi i\theta} \mapsto e^{2\pi in\theta}$. The span of $\{z_n : n \in \mathbb{Z}\}$ is called the space of trigonametric polynomials and will be denoted by $P(\mathbb{T})$. Just like the δ_n's these z_n's considered as elements of $L_2(\mathbb{T})$ will be denoted by the same symbol. From the context one has to make out the ambient space.

(1) c_{00} is dense in $\ell_2(\mathbb{Z})$.
(2) By the Stone-Weirstrass theorem $P(\mathbb{T})$ is dense in $C(\mathbb{T})$.
(3)

$$\langle z_n, z_m\rangle = \int_0^1 \overline{e^{2\pi in\theta}} e^{2\pi im\theta} d\theta = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

(4) $\mathcal{F}_{\mathbb{Z}} : \delta_n \mapsto z_{-n}$ for all $n \in \mathbb{Z}$.
(5) Combining the above we see that the Fourier transform restricted to c_{00} gives an innerproduct preserving bijective map from c_{00} to $P(\mathbb{T})$.

By proposition (1.11) $\mathcal{F}_{\mathbb{Z}}$ extends to a unitary. It remains to verify the intertwining condition (2.12).

$$\begin{aligned}
(\mathcal{F}_{\mathbb{Z}} \circ \pi_{\mathbb{Z}}(\delta_n))(\delta_k) &= \mathcal{F}_{\mathbb{Z}}(U(n)(\delta_k)) \\
&= \mathcal{F}_{\mathbb{Z}}(\delta_{k+n}) \\
&= z_{-(k+n)} \\
&= \widetilde{\pi}_{\mathbb{T}}(z_{-n})(z_{-k}) \\
&= \widetilde{\pi}_{\mathbb{T}}(\mathcal{F}_{\mathbb{Z}}(\delta_n))(z_{-k}) \\
&= (\widetilde{\pi}_{\mathbb{T}}(\mathcal{F}_{\mathbb{Z}}(\delta_n)) \circ \mathcal{F}_{\mathbb{Z}})(\delta_k).
\end{aligned}$$

Thus we have

$$(\mathcal{F}_{\mathbb{Z}} \circ \pi_{\mathbb{Z}}(\delta_n))(v) = (\widetilde{\pi}_{\mathbb{T}}(\mathcal{F}_{\mathbb{Z}}(\delta_n)) \circ \mathcal{F}_{\mathbb{Z}})(v), \forall v \in c_{00}.$$

Boundedness of the operators $(\mathcal{F}_{\mathbb{Z}} \circ \pi_{\mathbb{Z}}(\delta_n)), (\mathcal{F}_{\mathbb{Z}} \circ \pi_{\mathbb{Z}}(\delta_n))$ along with the density of c_{00} implies that

$$(\mathcal{F}_{\mathbb{Z}} \circ \pi_{\mathbb{Z}}(\delta_n)) = (\widetilde{\pi}_{\mathbb{T}}(\mathcal{F}_{\mathbb{Z}}(\delta_n)) \circ \mathcal{F}_{\mathbb{Z}}).$$

The inequalities (2.7,2.9) together with the density of c_{00} in $\ell_1(\mathbb{Z})$ completes the proof. □

Following the general prescription suggested in (2.10) the Fourier transform in the case of $G = \mathbb{T}$ is given by

$$\mathcal{F}_{\mathbb{T}} : L_1(\mathbb{T}) \to C_b(\mathbb{Z}); f \mapsto \hat{f}, \text{ where } \hat{f}(n) = \int_0^1 f(\theta)e^{-2\pi in\theta} d\theta.$$

Proposition 2.9 (Riemann-Lebesgue Lemma)**.** *Fourier transform of an integrable function on* $\mathbb{T}$ *vanishes at infinity, that is* $\hat{f}(n) \to 0$.

Proof. By the Stone-Weirstrass theorem trigonometric polynomials are dense in $C(\mathbb{T})$ and $C(\mathbb{T})$ is dense in $L_1(\mathbb{T})$. Hence trigonametric polynomials are dense in $L_1(\mathbb{T})$. So, given an $f \in L_1(\mathbb{T})$, we can find a trigonametric polynomial P such that $\|f - P\|_1 < \epsilon$. By (2.11) we know that

$$|(\hat{f} - \hat{P})(n)| \leq \|\mathcal{F}_{\mathbb{T}}(f - P)\| \leq \|f - P\|_1 < \epsilon.$$

By (2.13) we also know that $\hat{P}(n) = 0$ whenever $|n|$ is large say greater than some N. Therefore for n such that $|n| > N$, we have $|\hat{f}(n)| < \epsilon$. □

Theorem 2.10 (Unitarity of the Fourier transform)**.** *The Fourier transform $\mathcal{F}_{\mathbb{T}}$ restricted to $L_2(\mathbb{T})$ gives a unitary operator $\mathcal{F}_{\mathbb{T},2\to 2}$ from $L_2(\mathbb{T})$ to $\ell_2(\mathbb{Z})$.*

Proof. Let $f \in L_2(\mathbb{T}) \subseteq L_1(\mathbb{T})$. Note that $\hat{f}(n) = \langle z_n, f\rangle$ where $\{z_n : n \in \mathbb{Z}\}$ is the o.n.b of $L_2(\mathbb{T})$ given by $z_n(e^{2\pi i\theta}) = e^{2\pi in\theta}$. Now the proof follows from corollary (1.26) once we note that $\mathcal{F}_{\mathbb{T},2\to 2}$ is nothing but the unitary used in the proof of (1.26). □

Definition 2.11. *Given $f \in L_2(\mathbb{T})$ the n-th Fourier coefficient is defined as*

$$\hat{f}(n) = \langle z_n, f\rangle = \int_0^1 f(e^{2\pi i\theta})e^{-2\pi in\theta}d\theta$$

where $z_n : e^{2\pi i\theta} \mapsto e^{2\pi in\theta}$. The series of functions $\sum \hat{f}(n)e^{2\pi in\theta}$ is called the Fourier series of f.

Corollary 2.12. *Let $f \in L_2(\mathbb{T})$, then $\int_0^1 |f(e^{2\pi i\theta})|^2 d\theta = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.*

Proof.

$$\int_0^1 |f(e^{2\pi i\theta})|^2 d\theta = \|f\|^2 = \|\mathcal{F}_{\mathbb{T},2\to 2}(f)\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

□

3. Spectral Theorem for Compact Operators

Definition 3.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called compact if the image of the unit ball in $\mathcal{H}_1$ under T is precompact.*

These operators in general have a certain structure. In this section we will learn that. We begin with an alternative characterization of compact operators.

Proposition 3.2. *Let $T \in B(\mathcal{H})$ then T is compact if and only if T converts weakly convergent nets to norm convergent nets. That is*

$$(\langle v, u_\alpha\rangle \to \langle v, u\rangle, \forall v \in \mathcal{H}) \Longrightarrow \|T(u_\alpha) - T(u)\| \to 0.$$

Proof. Let $\{u_\alpha\}_{\alpha\in A}$ be a weakly convergent net with u as its limit. The net $\{T(u_\alpha)\}$ weakly converges to $T(u)$ because

$$\langle v, T(u_\alpha)\rangle = \langle T^*(v), u_\alpha\rangle \to \langle T^*(v), u\rangle\langle v, T(u)$$

In order to utilize the hypothesis that T is a compact operator note that the set $\{u_\alpha : \alpha \in A\}$ is weakly bounded. Hence by exercise (5.2) it is norm bounded. So there exists M such that $\sup\{\|u_\alpha\| : \alpha \in A\} < M$. Since T is compact any subnet of $\{T(u_\alpha)\}$ has a convergent subnet and the limit must be $T(u)$, because $\{T(u_\alpha)\}$ weakly converges to $T(u)$. Since the limit of the convergent subnet of any given subnet does not depend on the net the original net must be convergent with the same limit, i.e., $\|T(u_\alpha) - T(u)\| \to 0$.

Conversely, let $\{T(u_\alpha)\}$ be a net in $T(B(0,1))$. By Banach-Alaoglu theorem we can conclude that $\{u_\alpha\}$ has a convergent subnet. Then the corresponding subnet $\{T(u_\alpha)\}$ converges. This shows that $T(B(0,1))$ is relatively compact or equivalently has compact closure. □

Definition 3.3. *A linear operator is called finite rank if its range is a finite dimensional subspace.*

Theorem 3.4. *Let $T \in B(\mathcal{H})$, then T is a compact operator if and only if T is a norm limit of finite rank operators.*

Proof. **Only if part:** Let T be a compact operator. Therefore given $\epsilon > 0$, there exists $y_1, \cdots, y_{n_\epsilon}$ such that $\overline{T(B(0,1))} \subseteq \cup_{j=1}^{n_\epsilon} B(y_j, \epsilon)$. Let $\{e_\alpha\}_{\alpha \in A}$ be an o.n.b for $\mathcal{H}$. Then there exists a finite subset F of A such that

$$\sum_{\alpha \notin F} |\langle y_j, e_\alpha \rangle|^2 \quad < \quad \epsilon^2, \forall j = 1, \cdots, n_\epsilon. \tag{3.14}$$

Let y be an element of the norm closure of $T(B(0,1))$. Then there exists y_j such that that

$$\sum_{\alpha \notin F} |\langle y - y_j, e_\alpha \rangle|^2 \leq \sum_{\alpha \in A} |\langle y - y_j, e_\alpha \rangle|^2 = \|y - y_j\|^2 < \epsilon^2 \tag{3.15}$$

Therefore,

$$\sum_{\alpha \notin F} |\langle y, e_\alpha \rangle|^2 \quad < \quad 4\epsilon^2. \tag{3.16}$$

Let P_F be the orthogonal projection on the span of $\{e_\alpha : \alpha \in F\}$ and $T_F = P_F T$. By proposition 3.2 T_F is a compact operator. Let $x \in B(0,1)$ and $y = T(x) \in \overline{T(B(0,1))}$. By (3.16) we see that $\|T(x) - T_F(x)\| < 2\epsilon$. Therefore $\|T - T_F\| < 2\epsilon$.

If part: Let $\{T_n\}$ be a sequence of finite rank operators such that $\|T_n - T\| \to 0$. Let $\{u_\alpha\}$ be a weakly convergent net with u as its weak limit, i.e., $\langle v, u_\alpha \rangle \to \langle v, u \rangle, \forall v \in \mathcal{H}$. The set $\{u_\alpha\}$ is weakly bounded and hence by exercise (5.2) is norm bounded say by $M > 1$. Find N such that $\|T_n - T\| < \frac{\epsilon}{3M}$ whenever $n \geq N$. Let γ be such that $\|T_N u_\alpha - T_N u_\beta\| < \frac{\epsilon}{3M}$ provided $\alpha, \beta \succ \gamma$. Then for such α, β,

$$\|Tu_\alpha - Tu_\beta\| \leq \|Tu_\alpha - T_N u_\alpha\| + \|Tu_\beta - T_N u_\beta\| + \|T_N u_\alpha - T_N u_\beta\| \leq \epsilon.$$

Thus $\{T(u_\alpha)\}$ is a Cauchy net hence convergent. □

Theorem 3.5. *Let $T \in B(\mathcal{H})$ be a self-adjoint compact operator, $T \neq 0$, then*

$$\Lambda_+ = \sup\{\langle u, Tu\rangle : \|u\| = 1\} = \sup\{\langle u, Tu\rangle : \|u\| \leq 1\}$$

$$\Lambda_- = \inf\{\langle u, Tu\rangle : \|u\| = 1\} = \inf\{\langle u, Tu\rangle : \|u\| \leq 1\}$$

are attained. Let u_+, u_- b the vectors where Λ_+, Λ_- are attained, then at least one of the following holds,

$$Tu_\pm = \Lambda_\pm u_\pm.$$

Proof. Let $F(u) = \langle u, Tu\rangle$., then this is a real valued function because,

$$\overline{F(u)} = \langle Tu, u\rangle = \langle u, T^*u\rangle = \langle u, Tu\rangle = Fu).$$

Also for $\|u\| \leq 1$, $|F(u)| \leq \|u\|^2\|T\| \leq \|T\|$. Therefore $\Lambda_\pm$ makes sense. Let $\{u_n\}$ be a sequence such that $\|u_n\| \leq 1$ and $F(u_n) \to \Lambda_+$. Since a Hilbert space is reflexive by Banach-Alaoglu theorem its unit ball is weakly compact the sequence $\{u_n\}$ has a weakly convergent subsequence. Without loss of generality we can assume that $u_n \to u_+$, weakly. Then,

$$\begin{aligned}
|F(u_n) - F(u_+)| &= |\langle u_n, Tu_n\rangle - \langle u_+, Tu_+\rangle| \\
&\leq |\langle u_n, Tu_n - Tu_+\rangle| + |\langle u_n - u_+, Tu_+\rangle| \\
&\leq \|Tu_n - Tu_+\| + |\langle u_n - u_+, Tu_+\rangle| \\
&\to 0.
\end{aligned}$$

Since T is compact the first term goes to zero and the second term goes to zero because $\{u_n\}$ weakly converges to u_+. Therefore $F(u_+) = \lim F(u_n) = \Lambda_+$. Clearly $\|u_+\| = 1$, because otherwise there exists $\epsilon > 0$ such that $\|(1+\epsilon)u_+\| = 1$, and $F((1+\epsilon)u_+) = (1+\epsilon)F(u_+) > F(u_+)$. Similarly we obtain u_- such that $F(u_-) = \Lambda_-$.

$\Lambda_\pm$ *both can not be zero:* Suppose that $\Lambda_+ = \Lambda_- = 0$. Then for any u of unit norm, $F(u) = 0$. Thus for any u, we get $\langle u, Tu\rangle = 0$. Then by polarization we get

$$2\langle v, Tu\rangle = \langle u+v, T(u+v)\rangle + i\langle u+iv, T(u+iv)\rangle = 0.$$

Therefore $T = 0$ a contradiction to $T \neq 0$! □

Without loss of generality we assume that $\Lambda_+ \neq 0$. Then $\langle u_+, Tu_+\rangle = \Lambda_+ > 0$. Therefore, $T(u_+) \neq 0$.
Claim: $v \in \mathcal{H}, \|v\| = 1, v \perp u_+ \implies v \perp Tu_+$

Proof of Claim: Let $v_\theta = (Cos\theta)v + (Sin\theta)u_+$, then $\|v_\theta\| \leq 1$ and

$$\begin{aligned} F(v_\theta) &= Cos^2\theta.F(v) + Sin^2\theta.F(u_+) + Cos\theta Sin\theta\langle v, Tu_+\rangle \\ &\quad + Sin\theta Cos\theta\langle u_+, Tv\rangle \\ &= Cos^2\theta F(v) + Sin^2\theta F(u_+) + Sin2\theta\Re\langle v, Tu_+\rangle \end{aligned}$$

We know that the function $\theta \mapsto F(v_\theta)$ attains its maximum at $\theta = \pi/2$. Therefore

$$\frac{dF(v_\theta)}{d\theta}|_{\theta=\pi/2} = \Re\langle v, Tu_+\rangle = 0.$$

Instead of v if we put $\sqrt{-1}v$ we obtain $\Im\langle v, Tu_+\rangle = 0$. Therefore $\langle v, Tu_+\rangle = 0$. □

Thus, $Tu_+ \in {u_+^\perp}^\perp = \mathbb{C}u_+$. Let $Tu_+ = \lambda u_+$, and

$$\Lambda_+ = F(u_+) = \langle u_+, Tu_+\rangle = \lambda\|u_+\|^2 = \lambda.$$

If $\Lambda_- \neq 0$ we similarly conclude that $Tu_- = \Lambda_- u_-$. □

Lemma 3.6. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then*

$$\|T\| = \sup\{|\langle u, Tu\rangle| : \|u\| = 1\}. \tag{3.17}$$

Proof. Let M be the right hand side of 3.17. By Cauchy-Schwarz inequality we see that $M \leq \|T\|$. Let $u, v \in \mathcal{H}$ be unit vectors and $w = \frac{\langle Tv,u\rangle}{|\langle u,Tv\rangle|}v$. Then

$$\begin{aligned} \langle u, Tw\rangle &= |\langle u, Tv\rangle| \\ \langle Tw, u\rangle &= \overline{\langle u, Tw\rangle} = \langle u, Tw\rangle. \end{aligned}$$

If $u + w \neq 0$,

$$|\langle u + w, T(u + w)\rangle| = \|u + w\|^2 |\langle \frac{u + w}{\|u + w\|}, T(\frac{u + w}{\|u + w\|})\rangle| \leq M\|u + w\|^2.$$

If $u + w = 0$ anyway this inequality holds. Similarly for $u - w$ we have,

$$|\langle u - w, T(u - w)\rangle| \leq M\|u - w\|^2.$$

Note that

$$\begin{aligned} \langle u + w, T(u + w)\rangle &= \langle u, Tu\rangle + 2|\langle u, Tv\rangle| + \langle v, Tv\rangle| \\ \langle u - w, T(u - w)\rangle &= \langle u, Tu\rangle - 2|\langle u, Tv\rangle| + \langle v, Tv\rangle| \end{aligned}$$

Subtracting we get

$$\begin{aligned} 4|\langle u, Tv\rangle| &= \langle u+w, T(u+w)\rangle - \langle u-w, T(u-w)\rangle \\ &\leq |\langle u+w, T(u+w)\rangle| + |\langle u-w, T(u-w)\rangle| \\ &\leq M(\|u+w\|^2 + \|u-w\|^2) \\ &= 2M(\|u\|^2 + \|w\|^2) \\ &= 4M, \text{ since } \|u\| = \|w\| = 1. \end{aligned}$$

Therefore $|\langle u, Tv\rangle| \leq M$ and taking supremum over the left had side we obtain the desired inequality $\|T\| \leq M$. □

Notation: Given a pair of vectors $u, v \in \mathcal{H}$, $|u\rangle\langle v|$ stands for the operator $w \mapsto \langle v, w\rangle u$. In particular $P_u := |u\rangle\langle u|$ is the orthogonal projection onto the span of u.

Theorem 3.7 (Spectral Theorem for Compact Self-adjoint Operator)**.** *Let $T \neq 0$ be a compact self-adjoint operator on $\mathcal{H}$. Then there exists a sequence $\{\lambda_n\}$ of real numbers and a sequence of mutually orthogonal vectors $\{e_n\}$ such that $|\lambda_n| \to 0, \|e_n\| = 1 \forall n$ and*

$$T = \sum \lambda_n |e_n\rangle\langle e_n|, \tag{3.18}$$

where the sum appearing in (3.18) is norm convergent.

Proof. Let $T^{(0)} = T, \mathcal{H}^{(0)} = \mathcal{H}$. Now we will successively define

(1) Hilbert spaces $\mathcal{H}^{(n)}$ for $n \geq 0$ such that $\mathcal{H}^{(n+1)} \subseteq \mathcal{H}^{(n)}$.
(2) Compact self-adjoint operators $T^{(n)} : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$.
(3) Vectors $e_{n+1} \in \mathcal{H}^{(n)}$ orthogonal to $\mathcal{H}^{(n+1)}$ and scalars λ_{n+1} for $n \geq 0$.

This will be defined in a manner so that if $Q^{(n)}$ denotes the orthogonal projection onto $\mathcal{H}^{(n+1)}$ then

$$T^{(n+1)} = T^{(n)}Q^{(n)} = Q^{(n)}T^{(n)} \tag{3.19}$$

$$T^{(n)} = \lambda_{n+1}P_{e_{n+1}} + T^{(n+1)}, \text{ for } n \geq 0, \tag{3.20}$$

$$\|T^{(n+1)}\| \leq |\lambda_{n+1}|. \tag{3.21}$$

This is achieved through repeated applications of theorem (3.5). Assume that we have defined $(T^{(k)}, \mathcal{H}^{(k)})$ for $k \leq n$. If $T^{(n)} = 0$ then $T^{(n+1)} = 0, \mathcal{H}^{(n+1)} = \mathcal{H}^{(n)}, \lambda_{n+1} = 0, e_{n+1} = 0$. Otherwise apply theorem (3.5) for the operator $T^{(n)}$.

$$(\lambda_{n+1}, e_{n+1}) = \begin{cases} (\Lambda_+(T^{(n)}), u_+(T^{(n)})), \text{ if } \Lambda_+(T^{(n)}) \geq \Lambda_-(T^{(n)}) \\ (\Lambda_-(T^{(n)}), u_-(T^{(n)})) \text{ otherwise.} \end{cases}$$

Then $T^{(n)}e_{n+1} = \lambda_{n+1}e_{n+1}$ and consequently $\lambda_{n+1}P_{e_{n+1}} = T^{(n)}P_{e_{n+1}} = P_{e_{n+1}}T$. Let $Q^{(n)} = I_{\mathcal{H}^{(n)}} - P_{e_{n+1}}$ and $\mathcal{H}^{(n+1)}$ be the range of $Q^{(n)}$. If we take $T^{(n+1)} =$

$T^{(n)}Q^{(n)}$ then all the conditions will be met. Adding (3.20) for $0 \leq n \leq k$ we obtain,

$$T = \sum_{n=0}^{k} \lambda_{n+1} P_{e_{n+1}} + T^{(k+1)} \tag{3.22}$$

Since $\{e_n\}$ converges to zero weakly $|\lambda_n| = \|T(e_n)\|$ converges to zero. It follows from the inequality (3.21) that $\|T^{(n)}\|$ converges to zero. This proves (3.18). □

Definition 3.8. *Let $T \in B(\mathcal{H})$, then λ is an eigenvalue of T with eigenvector $u \neq 0$ if $Tu = \lambda u$. The subspace $E_\lambda = \{u \in \mathcal{H} : Tu = \lambda u\}$ is called the eigenspace corresponding to the eigenvalue λ.*

Corollary 3.9. *Let $T \neq 0$ be a compact operator with a spectral resolution given by (3.18). Then $\lambda \neq 0$ is an eigenvalue iff λ equals one of the λ_n's. Also*

$$E_\lambda = span\{e_n : \lambda_n = \lambda\}. \tag{3.23}$$

Proof. Let A be the orthonormal set consisting of e_n's. Extend it to an orthonormal basis A'. Let $\lambda \neq 0$ be an eigenvalue with eigenvector u. Then by corollary (1.22) $u = \sum_n \langle e_n, u\rangle e_n + \sum_{\alpha \in A' \setminus A} \langle \alpha, u\rangle \alpha$. Therefore $Tu = \sum_n \lambda_n \langle e_n, u\rangle e_n$. On the other hand $\lambda u = \sum_n \lambda \langle e_n, u\rangle e_n + \sum_{\alpha \in A' \setminus A} \langle \alpha, u\rangle \alpha$. Using $Tu = \lambda u$ we obtain,

$$\langle \alpha, u\rangle = 0, \forall \alpha \in A' \setminus A \tag{3.24}$$

$$\lambda\langle e_n, u\rangle = \lambda_n \langle e_n, u\rangle, \forall n. \tag{3.25}$$

Equation (3.24) tells us that u belongs to the closed linear span of e_n's. Hence there exists n_0 such that $\langle e_{n_0}, u\rangle \neq 0$. Using equation (3.25) for n_0 we conclude $\lambda = \lambda_{n_0}$. The converse is obvious.

Clearly the left hand side of (3.23) is contained in the right hand side and we need to show the other inclusion. Let $u \in E_\lambda$. Then not only u belongs to the span of e_n's, equation (3.25) tells us that $\langle e_n, u\rangle \neq 0$ only if $\lambda = \lambda_n$. Therefore using (1.5) we get,

$$u = \sum_{n:\lambda_n = \lambda} \langle e_n, u\rangle e_n \in span\{e_n : \lambda_n = \lambda\}.$$

□

4. Appendix

4.1. **Nets.** A partially ordered set $\mathcal{P}$ is a set along with an order relation $a \preceq b$ satisfying (i) $a \preceq a$ (reflexive), (ii) $a \preceq b, b \preceq a \implies a = b$ (antisymmetric) (iii) $a \preceq b, b \preceq c \implies a \preceq c$, (transitive). A directed set is a preordered (reflexive: $a \preceq a$, transitive: $a \preceq b \preceq c \implies a \preceq c$ordered set $\mathcal{P}$ such that for any two elements $a, b \in \mathcal{P}, \exists c \in \mathcal{P}$ such that $a \preceq c, b \preceq c$. A net in a topological space X is a map $x : \mathcal{P} \to X$ where $\mathcal{P}$ is a directed set. Nets are often denoted as $\{x_\alpha\}_{\alpha \in \mathcal{P}}$. A net $\{x_\alpha\}$ converges to $x \in X$ if given any open neighborhood U of x there exists α_0 such that for all $\alpha_0 \preceq \alpha, x_\alpha \in U$. A net $\{x_\alpha\}$ in a metric space (X, d) is called Cauchy if $\forall \epsilon > 0, \exists \alpha_0$ such that $d(x_\alpha, x_\beta) < \epsilon$ whenever $\alpha, \beta \succeq \alpha_0$.

Proposition 4.1. *Let X be a topological space and $U \subseteq X$. Then $x \in \overline{U}$ iff there exists a net x_α in U converging to x.*

Proof. Let x_α be a net converging to x. Then given any open neighborhood of x it will contain some x_α and hence an element from U. Thus x is contained in the closure of U. Conversely suppose x is an element from closure of U. Let $\mathcal{P}$ be the directed set consisting of open neighborhoods of x. Define $U_1 \preceq U_2$ if $U_2 \subseteq U_1$. Since x lies in the closure of U every neighborhood V of x will contain an element of U say x_V. This defines a net converging to x. □

Proposition 4.2. *In a complete metric space (X, d) every Cauchy net converges.*

Proof. Exercise (5.1) □

4.2. **Sums.** Let X be a set and $a : X \to \mathbb{C}$ be a function. We want to attach a meaning to $\sum_{x \in X} a(x)$. Consider the set $\mathcal{D}$ of finite subsets of X. Given two such finite subsets α, β of X we define $\alpha \preceq \beta$ if $\alpha \subseteq \beta$. With this order $\mathcal{D}$ becomes a directed set. Now consider the net $\{a_\alpha\}_{\alpha \in \mathcal{D}}$, where $a_\alpha = \sum_{x \in \alpha} a(x)$.If this net converges we say that the sum $\sum_{x \in X} a(x)$ is meaningful and $\sum_{x \in X} a(x) = \lim_{\alpha \in \mathcal{D}} a_\alpha$.

Proposition 4.3. *Let $a : X \to \mathbb{C}$ be such that $\sum_{x \in X} a(x)$ makes sense. Then $Supp(a) = \{x \in X : a(x) \neq 0\}$ is countable, and if we fix a one to one and onto map $\phi : \mathbb{N} \to Supp(a)$, then $\sum_{n=1}^{\infty} |a(\phi(n))| < \infty$ and $\sum_{x \in X} a(x) = \sum_{n=1}^{\infty} a(\phi(n))$. Note that this sum does not depend on the map ϕ.*

Proof. Let $\lim_{\alpha \in \mathcal{D}} a_\alpha = A$, that means given $\epsilon > 0$ there exists α_0 such that whenever we have a finite subset α of X such that

$$\alpha \supseteq \alpha_0, |a_\alpha - A| < \epsilon. \tag{4.26}$$

Let $X_n = \{x \in X : \Re(a(x)) > 1/n\}$, then X_n must be finite. Because otherwise, if we take a subset α_k of X_n of size kn, then $\Re(a_{\alpha_k}) > k$. Let $\beta_k = \alpha_k \cup \alpha_0$, then $\Re(a_{\beta_k}) = \Re(a_{\alpha_k}) + \Re(a_{\alpha_0}) > k + \Re(A) - \epsilon$. On the other hand by 4.26, $\Re(a_{\beta_k}) <$

$\Re(A) + \epsilon$, a contradiction. Similarly one shows that $\{x \in X : \Re(a(x)) < -1/n\}$ is finite. Thus we get $\{x \in X : |\Re(a(x)| > 1/n\}$ is finite. Exactly along the same lines one shows that $\{x \in X : |\Im(a(x))| > 1/n\}$ is finite. □

Remark 4.4. *Note that there is nothing special about the Banach space* $\mathbb{C}$*, if* E *is a Banach space and* $a : X \to E$ *is a function we can similarly define* $\sum_{x \in X} a(x)$

5. Exercises

Exercise 5.1. *In a complete metric space (X, d) every Cauchy net converges.*

Exercise 5.2. *Let E be a Banach space. Let X be a weakly bounded subset of E. That means for all $\phi \in E^*$, $\phi(X)$ is a bounded subset of $\mathbb{K}$. Then X is a norm bounded subset of E.*

Exercise 5.3 (Hermite Polynomials)**.** (1) $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$.

(2) *Let* $\rho_N(t,x) = e^{-\frac{(t^2-2tx)}{2}}$*,then show that*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho_N(t,x)\rho_N(s,x) e^{-\frac{x^2}{2}}\, dx = e^{ts}.$$

(3) *Let* $\rho_N(t,x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$*, show that*

$$\frac{1}{\sqrt{2\pi}} \int \frac{H_k(x)}{\sqrt{k!}} \frac{H_l(x)}{\sqrt{l!}} e^{-\frac{x^2}{2}}\, dx = \delta_{kl}, k, l = 0, 1, \cdots .$$

(4) $H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$.

These are orthogonal polynomials with respect to the measure $\mu(E) = \frac{1}{\sqrt{2\pi}} \int_E e^{-\frac{x^2}{2}}\, dx$.

Exercise 5.4 (Laguerre Polynomials)**.** *Let*

$$f_\Gamma(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \text{ for } x > 0 \\ 0, \text{ otherwise} \end{cases},$$

where $\alpha > 0$. Let μ be the measure given by $\mu(E) = \int_E f_\Gamma(x) dx$.

(1) *Let* $\rho_\Gamma(t,x) = \frac{1}{(1-t)^\alpha} e^{-\frac{tx}{(1-t)}}$*,then show that*

$$\begin{aligned} \int_0^\infty \rho_\Gamma(t,x)\rho_\Gamma(s,x) f_\Gamma(x) dx &= \frac{1}{(1-ts)^\alpha}, \\ &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} t^n s^n \end{aligned}$$

(2) *Let* $\rho_\Gamma(t,x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x)$*, show that*

$$\int \ell_k(x)\ell_l(x) f_\Gamma(x) dx = \delta_{kl}, k, l = 0, 1, \cdots .$$

where

$$\ell_k(x) = \left(\frac{1}{k!\alpha(\alpha+1)\cdots(\alpha+k-1)} \right)^{\frac{1}{2}} L_k(x).$$

Exercise 5.5. *Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ be a sesquilinear form. If there exists a positive constant C such that*

$$|T(u,v)| \le C\|u\|\|v\|, \forall u, v \in \mathcal{H}.$$

Then there is a unique bounded linear map $\widetilde{T} \in B(\mathcal{H})$ such that $\|\widetilde{T}\| \le C$ and

$$T(u,v) = \langle \widetilde{T}(u), v \rangle, \forall u, v \in \mathcal{H}.$$

Exercise 5.6. *If we have Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, and a sesquilinear map $B : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{K}$ such that*

$$|B(u,v)| \leq C\|u\|\|v\|, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2$$

where C is a positive constant then there exists a bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ of norm less than or equal to C and

$$B(u,v) = \langle T(u), v\rangle, \forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2.$$

Exercise 5.7. *Let $T \in B(\mathcal{H})$. Then there is a unique linear map denoted by T^* such that*

$$\langle T^*(u), v\rangle = \langle u, T(v)\rangle, \quad \forall u, v \in \mathcal{H}. \tag{5.27}$$

Moreover $\|T^*\| = \|T\|$ and $T^{**} = T$.

Exercise 5.8. *In the set up of exercise (5.5) there exists a unique bounded linear map $T' \in B(\mathcal{H})$ such that $\|T'\| \leq C$ and*

$$T(u,v) = \langle u, T'(v)\rangle, \forall u, v \in \mathcal{H}.$$

Exercise 5.9. *A bounded linear map U on a Hilbert space is a unitary iff $U^*U = UU^* = I$, where I stands for the identity operator.*

Exercise 5.10 (Lax-Milgram)**.** *The bilinear form T is called coercive if $\exists a > 0$ such that $T(u,u) \geq a\|u\|^2, \forall u \in \mathcal{H}$. By exercise (5.5) we know that there exists $\widetilde{T} \in B(\mathcal{H})$ such that $T(u,v) = \langle \widetilde{T}(u), v\rangle$. If T is given to be coercive.*

(i) *Show that $\widetilde{T}$ is one to one.*
(ii) *Let $\mathfrak{Ran}$ be the range of $\widetilde{T}$. Consider $S : \mathfrak{Ran} \to \mathcal{H}$ given by $S(u) = v$ where $u = \widetilde{T}(v)$. Show that S is bounded. and using this show that $\mathfrak{Ran}$ is closed.*
(iii) *Show that $\widetilde{T}$ is onto i.e., $\mathfrak{Ran} = \mathcal{H}$.*
(iv) *Conclude given $\phi \in \mathcal{H}$ there exists unique $u \in \mathcal{H}$ such that $T(u,v) = \langle \phi, v\rangle, \forall v \in \mathcal{H}$.*

Exercise 5.11. *Let $x, y : [0,1] \to \mathbb{R}$ be C^1-functions such that $\|\frac{dx}{dt}\|^2 + \|\frac{dy}{dt}\|^2 = \ell^2$, then $|\int_0^1 y(t)\frac{dx}{dt}dt| \leq \frac{\ell^2}{4\pi}$.*

Exercise 5.12. *Let $(\Omega, \mathfrak{S}, \mu)$ be a probability space and $\mathfrak{S}' \subseteq \mathfrak{S}$ a sub-σ-algebra. Let f be a nonnegative measurable L_1 function. Let $L_2(\mathfrak{S}')$ be the space of square integrable $\mathfrak{S}'$ measurable functions. Then $L_2(\mathfrak{S}') \subseteq L_2(\mathfrak{S})$ is a closed subspace. Let P be the corresponding projection. Show that*

(1) *If $0 \leq f \leq C$ then $\exists N \in \mathfrak{S}', \mu(N) = 0$ and a $\mathfrak{S}'$ measurable g such that on $N^c, 0 \leq g \leq C$ and $g = Pf$ a.e. Such a g will be called a version of Pf.*
(2)

$$\int_A f d\mu = \int_A Pf d\mu, \forall A \in \mathfrak{S}'. \tag{5.28}$$

(3) *Let* $f_n = f \wedge n$, *then* $\exists N \in \mathfrak{S}', \mu(N) = 0$ *such that outside* N, *each* Pf_n *has a version* g_n *such that* $0 \leq g_n \leq n$ *and* $g_n \leq g_{n+1}, \forall n \geq 1$. *Let* $g = \lim g_n$. *Show that*

$$\int_A f d\mu = \int_A g d\mu, \forall A \in \mathfrak{S}'. \tag{5.29}$$

Such a g *is called the conditional expectation of* f *given* $\mathfrak{S}'$ *and is denoted by* $\mathbb{E}(f|\mathfrak{S}')$. *This is an* $\mathfrak{S}'$ *measurable integrable function unique upto a* μ *null set.*